Sequential Search Auctions with a Deadline

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Sequential Search Auctions with a Deadline[*]

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A seller wants to allocate an indivisible product among a number of potential buyers by a finite deadline, and to contact a buyer, she needs to pay a positive search cost. We investigate the optimal mechanism for this problem, and show that its outcomes can be implemented by a sequence of second-price auctions. The optimal sequential search auction is characterized by declining reserve prices and increasing search intensities (sample sizes) over time, and the monotonicity results are robust in both cases of short-lived and long-lived bidders. When bidders are long-lived the optimal reserve prices demonstrate a one-step-ahead property, and our results generalize the well-known results in sequential search problems (Weitzman, 1979). We further examine an efficient search mechanism, and show that it is featured by both lower reserve prices and search intensities than an optimal search mechanism.
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It is puzzling to observe that many important selling processes in markets seem not competitive, where no obvious competition among buyers is observed. For instance, in mergers and acquisitions (M&As), it is a well-documented fact that the dominant selling process is one-on-one negotiation. That is, when the board of directors decides to sell a firm, in most cases, they just contact one potential buyer. Betton, Eckbo and Thorburn (2008) report that 95% of their sample deals in the US market, during the period from 1980 to 2005, are classified as non-competitive negotiations, and Andrade, Mitchell and Stafford (2001) also describe the prototypical M&As in the 1990s as friendly transactions, where normally there was just one bidder. This is against intuition, as conventional wisdom states that competition among bidders can not only raise bid premiums, but promote allocative efficiency in markets.

There have been some explanations for this puzzle. Boone and Mulherin (2007) present a new measure of M&A competition, and show that many deals classified as negotiations are actually auctions, where more than one bidders get involved in the competition. After reconstructing a new sample using their measure, however, they still have

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half of the sample deals classified as non-competitive negotiations. On the other hand, Aktas, De Bodt and Roll (2010) argue that M&A involves a seller's sequential decision, and a negotiation in an early stage is under the threat of following-up auctions. For instance, if the seller fails to achieve a good deal in the negotiation stage, she may invite more bidders and run auctions among them in the following stages. Therefore, a one-on-one negotiation is not insulated from competition indeed, and the pressure of following-up auctions can drive up bid premiums in the negotiation stage. This argument is supported by the empirical evidence that there is no significant difference in bid premiums across the two selling processes of negotiation and auction (Boone and Mulherin, 2007, 2008).

Depending on how many bidders to contact in the first stage, Boone and Mulherin (2009) classify the selling processes in M&As into three categories: *one-on-one negotiation*, where a seller approaches a single most likely buyer first; *private controlled sale* (auction), where a seller screens and first invites a small number of qualified bidders to an auction; and *public full-scale auction*, where a seller announces and runs a public simultaneous auction, and all interested bidders can submit bids. Besides the nature of sequential decisions, a typical M&A selling process also involves a finite deadline for completing the transaction, and a positive search cost for a seller to contact bidders, i.e., it could be the seller's information costs due to the loss of proprietary information to bidders in a due diligence process.

In fact, M&A can be thought of as an example of the following general problem. A seller wants to allocate an indivisible product among a number of potential bidders; to contact a bidder, she needs to incur a search cost; and the seller has to complete the transaction by a finite deadline. Many important transactional processes can be thought of as a variant of this problem, such as matching in marriage markets with age deadlines, academic recruitment in the UK by a REF deadline, sequential talent contests, and so on. We are interested in the optimal allocation mechanism and its implementation in practice.

In this paper, we develop a framework for understanding the optimal choice of various selling processes in M&As and other similar problems. Specifically, we model it as a seller's sequential search problem with a finite deadline. Due to the presence of search costs, a simultaneous full-scale auction is typically not optimal. For example, if a seller searches bidders sequentially and happens to get an ideal offer from a bidder, she may not still have incentives to contact other bidders, as search is costly. Second, due to the presence of a finite deadline, a one-by-one sequential search may not be optimal either, as too few bidders might be sampled in this case. Therefore, when facing a finite deadline, a seller may conduct a compound search in which she searches both sequentially and simultaneously, i.e., she may sample several bidders simultaneously in a single period. We show that the optimal search mechanism, in this case, is featured by declining reserve prices and increasing search intensities (sample sizes). Our result of increasing search intensities may explain why, in M&As and other similar problems, the dominant selling process is non-competitive negotiations, rather than full scale auctions.

In our model, a (female) seller wants to allocate an indivisible product among a set N

of potential (male) bidders. To contact a bidder, she needs to pay a positive search cost, and she has to complete the transaction within a finite T periods. A search mechanism is composed of both a sampling rule and a sequence of stage mechanisms. The sampling rule specifies a sequence of bidder samples that the seller will contact in each period, while a stage mechanism defines the allocation and payment rules for the transaction in each period. If a product is not allocated in one period according to the stage mechanism, then the seller moves on to the next period, e.g., continue searching, until the deadline.

We assume the seller first announces the search mechanism, and then fully commits to it thereafter. The standard results in mechanism design show that when the search mechanism is *incentive feasible*, a bidder's expected payment is equal to his virtual value (Myerson, 1981), and therefore, the seller's expected revenue from a sample of bidders is equal to the highest virtual value of those bidders. With this result, we could conveniently transform a compound search problem into a sequential one. For instance, we can think of a sample of bidders as an aggregate bidder, who is characterized by the highest virtual value and the total search costs of those bidders.

We consider both cases of *long-lived* and *short-lived* bidders in our model. A long-lived bidder, once invited, will stay in the transaction thereafter until the end of period T. A seller then can reclaim a previously declined bid without the need of paying extra search cost. The case of long-lived bidders is analogous to sequential search with *full* recall. In contrast, a short-lived bidder will participate in the transaction only once, when he is invited, and then goes away. A declined bid of a short-lived bidder can never be reclaimed by the seller in later periods. The case of short-lived bidders corresponds to sequential search with *no* recall.

We show that, in both cases, the outcomes of an optimal search mechanism can be implemented by a sequence of second price auctions. Specifically, the rule of the sequential auction is as follows: the seller first invites a set M_1 of bidders to an auction with reserve price r_1; if any bidder submits an effective bid, then the transaction ends, and the payment and allocation are implemented according to the auction rule; if no bidder submits an effective bid, then the seller moves on to the next period and invites a set M_2 of new bidders to the auction, with a new reserve price r_2; the seller then continues with this process until the end of period T. If bidders are long-lived, all previously invited bidders will participate in the stage auctions of the following periods. On the contrary, short-lived bidders just participate in the stage auction when they are invited.

Our model generates several interesting results. Firstly, we show that an optimal sequential search auction is characterized by decreasing reserve prices and increasing search intensities over time, and the monotonicity results are robust in both cases of long-lived and short-lived bidders. The result of decreasing reserve prices is known in the literature of sequential search and sequential auctions. The intuition is that the value of optimal reserve reflects the continuation value of following an optimal search procedure in the remaining periods, which becomes smaller when the deadline approaches. As a result, when following an optimal search procedure, the reserve prices for stage auctions will keep on declining and reaches its lowest level in the last period.

The result of increasing search intensities is more interesting. When bidders are ex-

ante homogeneous, the degree of search intensity is simply measured by the number of bidders sampled in each period. Our result states that, when following an optimal search procedure, a seller will invite increasingly more bidders in each period when the deadline approaches. In another word, in the first period, the sell will contact the fewest number of bidders. This result of increasing search intensities helps explain why, in M&As and other similar problems, the dominant selling process could be non-competitive negotiations.

Secondly, we show that when bidders are long-lived, the optimal reserve price demonstrates a one-step-ahead property, in the sense that it just depends on the sample of bidders to be invited in the right next period, not further. This property is implied by the fact that optimal reserve prices are declining over time and search is with full recall. Our result on optimal reserve prices generalizes the well-known result of Weitzman (1979), which studies one-by-one sequential search without a deadline. Our model differs from his in at least two aspects: first, in our model, the targets for search are strategic bidders, while in his paper, they are non-strategic boxes containing random prizes;[1] second, we study compound search, where a seller searches both sequentially and simultaneously, yet Weitzman just considered the case of one-by-one sequential search. Our result of optimal reserve prices incorporates the well-known formula of Weitzman (1979) as a special case, and the result, we believe, can be applied to the studies of a large variety of related problems. In addition, we provide a succinct formula for maximum expected profit, when a seller follows an optimal search procedure. When T converges to infinity and bidders are homogeneous, our problem converges to a stationary and infinite-horizon (SIH) search problem. Our formula also takes the well-known result on maximum search profit in SIH search problems as a special case.

Thirdly, we examine an efficient sequential search mechanism when there is a finite deadline. We show that an efficient search mechanism is also featured by decreasing reserve prices and increasing search intensities over time. Interestingly, a simple comparative result shows that an efficient search mechanism has both lower reserve prices (when bidder samples given) and search intensities (when cutoff values are given) than in the corresponding optimal mechanism. Our result also indicates that, the the context of sequential auctions, the inefficiency of an optimal mechanism can result from an inefficient search procedure.

Finally, we compare the optimal search auctions across the two cases of long-lived and short-lived bidders. We first show that, for a given sampling rule, the optimal reserve prices for short-lived bidders are lower than those for long-lived bidders in each period. This result is intuitive, as in the case of short-lived bidders, a seller's fall-back revenue is always zero, smaller than that for long-lived bidders, and therefore the seller is willing to accept a lower reserve price. Second, we show that, in the case of short-lived bidders, the optimal reserve prices no longer hold the one-step-ahead property, yet we are able to provide a recurrence equation for optimal cutoff values.

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[1]Crémer, Spiegel and Zheng (2007) also consider sequential search in an auction contexts, where a seller needs to incur positive costs to search bidders. Like Weitzman (1979), but they focus solely on infinite sequential search with full recall, as bidders are assumed to be long-lived in their model. In this paper, we investigate sequential search with a deadline, and consider both cases of long-lived and short-lived bidders.

The rest of this paper is organized as follows. Section I provides a brief review of the related literature. Section II setups the basic model, where we also define the search mechanism and characterize the optimal search rule. Section III proposes a sequential search auction that can implement the outcomes of the optimal search mechanism. Section IV studies an efficient search mechanism, and compares its outcomes with the optimal search mechanism. Section V moves on to the other important case of short-lived bidders and characterizes the optimal search mechanism in this case. Section VI is a short conclusion. Missing proofs appear in Appendix.

I. Related Literature

This study is motivated by the puzzling observation that many important selling processes in markets seem non-competitive, where no obvious competition among buyers is observed. We propose a new framework for understanding these puzzles, by modeling them as a finite sequential search problem for a seller. First, our paper fills an important gap in the theory literature on sequential search and auctions, and it is related to the following strands of literature: (1) sequential search without or with a deadline, (2) sequential auctions with participation costs, and (3) auctions with buy-price options. Second, our paper also makes practical contributions to the literature, and it is related to the persistent debate in practice on the choice among various selling processes in markets.

For the literature on search, Weitzman (1979) is a seminal paper that studies the so-called Pandora's problem of infinite sequential search with *full* recall. Specifically, Pandora faces a number of closed boxes; insides each box, there is a random prize; she needs to pay a positive search cost to open a box; Pandora can open just one box in a single period, and her objective is to maximize the expected value of the prize discovered, net of the total search cost. Weitzman provides a nice solution to this problem, which is known as Pandora's Rule. First, Pandora can derive a cutoff prize for each box, at which she is indifferent between keeping the cutoff prize and inspecting that box at a cost. Second, the selection rule specifies that if Pandora intends to open a box, it should be the box with the highest cutoff prize among all the remaining unopened boxes. Third, the stopping rule suggests that Pandora should stop searching whenever the highest discovered prize so far is greater than the highest cutoff prize of all the remaining unopened boxes. Pandora's rule then implies that, under an optimal search procedure, the cutoff prizes are necessarily declining over time.

Crémer, Spiegel and Zheng (2007) extend the sequential search model of Weitzman (1979) into an auction context.[2] Instead of opening boxes, they consider a seller inviting bidders sequentially at positive search costs, and also asking their bids sequentially. They just consider the case of long-lived bidders, where a seller can reclaim previously

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[2]McAfee and McMillan (1988) precede Crémer, Spiegel and Zheng (2007) in considering sequential search in an auction context. In their model, a monopolist buyer seeks to buy an indivisible product from one of a set of producers. The buyer contacts the producers sequentially, each at a constant cost, and the producers are ex-ante homogeneous. They show that the optimal search mechanism, in their model of homogeneous producers, is a combination of *constant* reservation-price search and auction.

declined bids. As mentioned earlier, this corresponds to search with full recall. They show that Pandora's rule is still valid in this case, and further compare the outcomes between optimal and efficient search auctions.

In the presence of a finite deadline, a one-by-one sequential search may no longer be optimal. In this case, Pandora may intend to sample multiple boxes simultaneously in a single period. This more general search procedure is known as a compound or sequential-and-simultaneous search. Gal, Landsberger and Levykson (1981) is an early paper that studies this compound search procedure in labor markets. They introduce a finite deadline into the classic sequential search model of Lippman and McCall (1976). In their model, job offers are homogeneous in terms of search cost and wage distribution, and the number of offers sampled in a single period thus measures search intensity. They show that, when search is with no recall, a searcher's optimal search procedure is featured by decreasing reservation wages and increasing search intensities over time. In another word, when the deadline approaches, a job searcher will invest more and more in search.[3]

Morgan (1983) examines a similar model to that of Gal, Landsberger and Levykson (1981), and provides more results. First, he also shows that, when search is with no recall, the optimal sample size is increasing in time. Second, when a search is with full recall, he provides a sufficient condition under which the optimal sample size is still increasing in time, yet this result does not hold in general. Morgan and Manning (1985) further present some results on the existence and properties of optimal search rules for compound search problems, when a searcher can choose both the sample size in each period and the number of periods she may engage in search.

Our paper fills a critical gap in the literature, by investigating finite sequential search in an auction context. Compared with other papers studying sequential search auctions (McAfee and McMillan, 1988; Crémer, Spiegel and Zheng, 2007), our paper differs in at least two aspects. First, we study sequential search with a deadline, where a seller may conduct a compound search, and it takes the one-by-one sequential search in the earlier studies as a special case. Second, we investigate both cases of long-lived and short-lived bidders in this paper, while the other studies focus solely on long-lived bidders.

Besides the contribution to theoretical literature, our study is also of practical importance. Many transaction processes in markets, such as M&As, matching in marriage markets, job recruitment, and among others, can be fitted into our framework. Particularly, our paper is related to the persistent debate on the choice of various selling mechanisms, e.g., between negotiation and auction, in markets.

Bulow and Klemperer (1996) show that an English auction with $n+1$ bidders yields strictly higher revenue than an optimal auction with n bidders, and therefore, the value of bargaining power is bounded above by the value of competition by inviting one more bidder. In a recent paper (Bulow and Klemperer, 2009), they compare the selling mechanisms of a sequential negotiation and a simultaneous auction. They show that a simultaneous auction yields higher revenue for a seller than a sequential negotiation, though the

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[3]Benhabib and Bull (1983) also study search intensity in job markets, where job offers are homogeneous and a job seeker can not reclaim a previously rejected offer. They show that in an optimal search, the reservation wage is decreasing in time, while the search intensity (sample size in each period) is increasing in time. They also consider the on-the-job search, and derive some different results.

latter is always more efficient as more bidder information is exploited. In their model, bidders need to pay positive entry costs to participate in the transaction, and a seller is unable to commit to a take-it-or-leave-it offer. Under sequential negotiation, an already entered bidder can make a jump-bid so as to deter further entry of outside bidders, which may harm the seller. As a result, a seller usually prefers a simultaneous auction to a sequential negotiation.

But the empirical evidence seems not supporting their results. For example, in M&As, it is one-on-one negotiation, not competitive auction, that is the dominant selling process. The empirical evidence also shows that there is no significant difference in bid premiums across the two selling mechanisms of negotiation and auction in M&As (Boone and Mulherin, 2007, 2009). In this paper, we propose a different theory and model the selling process as a finite sequential search problem for a seller. We show the robust result that, under optimal search procedure, a seller's search intensity is increasing over time, and therefore she will contact the fewest number of buyers in the first stage. Our theory helps explain the seeming absence of competition in M&As and other similar selling processes.

Third, our paper is also related to the literature on buy-price auctions. Reynolds and Wooders (2009) show that, when bidders are risk-averse, a buy-price then auction mechanism can generate higher revenue than an auction mechanism, as the presence of a buy-price option reduces bidders' risk-premium. In a similar model, Liu et al. (2017) investigate and justify the online buy-price auctions as a valid selling mechanism. In a recent paper, Zhang (2017) studies the optimal sequence of posted-price and auction in a sequential mechanism, where running an auction is more costly for a seller. In his model, a population of short-lived bidders enters the market periodically, and in each period, the seller chooses between a posted-price and an auction mechanism. He shows that, when the good has to be sold before a deadline and the auction cost is moderate, the optimal mechanism sequence takes the form of posted-prices then auctions.

Finally, our paper is related to the growing literature on sequential auctions and revenue management. Skreta (2015) investigates optimal sequential auctions with limited commitment, where revelation principle is no longer applicable. She shows that, in the case of no commitment, the optimal mechanism is qualitatively similar to one under full commitment, e.g., with descending reserve prices. In her paper, the same population of bidders participate in each round of the auction, while in our model, the population of bidders changes over time, which may be more realistic in M&As and other related situations. Said (2011, 2012) studies sequential auctions of multi-unit product with changing population, yet in a different environment to our model. Liu et al. (2017) study sequential auctions in the case of limited commitment. Other recent literature on revenue management includes Board and Skrzypacz (2016) with forward-looking buyers, in the case of full commitment, and Dilme and Li (2017), who study revenue management with the arrivals of strategic buyers in the case of no commitment. In our paper, we study a sequential search auction under the assumption of full commitment.

II. The Model

A. Model Setup

A (female) seller wants to allocate an indivisible product among a set $N = \{1, 2, \cdots, n\}$ of potential (male) bidders within T periods. Bidder i's value of the product, denoted by V_i, is distributed according to distribution F on $[0, 1]$, with strictly positive density $f > 0$ on $(0, 1)$, and V_i's are independent across bidders. We assume that F is of increasing failure rate (IFR), which implies the virtual value function of $\psi(v) = v - [1 - F(v)]/f(v)$ is strictly increasing in v. The distribution of F is common knowledge, yet the realization of V_i is only observed by bidder i.

To invite bidder i to participate in the transaction, the seller needs to incur a non-refundable search cost of $c_i \geq 0$.[4] A bidder can not submit a bid if not invited. The seller is a profit maximizer, and her objective is to maximize the expected product revenue, net of gross search costs. We normalize the seller's value of the product to 0. All the players are risk-neutral, and there is no time discounting.

The following assumption shows that the search cost c_i is small enough, such that all the bidders are valuable for the seller, i.e., for all $i \in N$,

$$(1) \qquad \int_{r^*}^{1} \psi(x)\, dF(x) > c_i,$$

where $\psi(r^*) = 0$. The left hand side of the inequality is just the maximum expected revenue the seller can obtain from a truthful bidder, and it is assumed to be greater than the search cost of c_i for all $i \in N$.

We assume bidders are long-lived in this section. A long-lived bidder, once invited, will stay in the transaction till the end of period T. This assumption also implies that the seller can reclaim a previously declined bid of a bidder in later periods, without the need of paying extra search cost. Under the assumption of long-lived bidders, the seller's search problem is analogous to the case of sequential search with *full* recall. Later in Section 6, we will turn to the other important case of short-lived bidders, which corresponds to sequential search with *no* recall.

B. A Compound Search Mechanism

Due to the presence of positive search costs, a simultaneous search mechanism, where a seller invites several bidders simultaneously to participate in a spot transaction, is usually not optimal. Similarly, due to the presence of a deadline, a one-by-one sequential search mechanism may not be optimal either. Here we consider a more general compound search mechanism, where a seller searches both sequentially and simultaneously, i.e., she may contact multiple bidders simultaneously in one period. This compound

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[4]There are several possible interpretations for the presence of a positive search cost. For instance, in M&As, it could be the information cost paid by the target firm (the seller), due to the loss of its proprietary information to potential acquirers (bidders) in the due diligence process, as the acquirers can be its direct competitors in the same market.

mechanism incorporates the simultaneous search mechanism and the sequential search mechanism as its special cases.

The compound search mechanism is a combination of a sampling rule and a sequence of stage mechanisms. For the sampling rule, we first define $\mathbf{M}^T = \left\{ M^1, M^2, \cdots, M^T \right\}$ as a family of disjoint subsets of N such that $M^j \cap M^{j'} = \emptyset$ for any $j \neq j'$, and $\bigcup_{j=1}^T M^j \subseteq N$. A sampling rule of the seller is then a permutation of the set $\mathbf{M}^T$, denoted by $\mathbf{M} = \{ M_1, M_2, \cdots, M_T \}$, which specifies an ordered sequence of bidder samples that a seller will search in each period. For instance, if $M_t = M^j$, then the seller will invite the sample M^j of bidders in period t. We further denote $N_t = \bigcup_{\tau=0}^t M_\tau$ as the set of bidder samples the seller has sampled until the end of period t, with $N_0 = \emptyset$, and $N_t^c = \mathbf{M}^T \backslash N_t$ denotes the set of M^j's that the seller has yet sampled at that point.

The other component is a sequence of stage mechanisms, that specifies a pair of allocation rule of $\mathbf{Q}_t$ and payment rule of $\mathbf{P}_t$ for each period. Specifically, for given a sampling rule of $\mathbf{M}$, the seller offers a stage mechanism of $(\mathbf{Q}_t, \mathbf{P}_t)$ to the set N_t of bidders in period t. Remember, under our assumption of long-lived bidders, there will be an accumulated set N_t of bidders participating the stage transaction in period t. The allocation rule of $\mathbf{Q}_t$ and the payment rule of $\mathbf{P}_t$ are respectively a mapping from the N_t bidders' revealed values to the allocation probabilities and the corresponding payments made by the N_t bidders. The sequence of the stage mechanisms is denoted by $(\mathbf{Q}, \mathbf{P}) = \left\{ (\mathbf{Q}_t, \mathbf{P}_t) \right\}_{1 \leq t \leq T}$.

The compound search mechanism, denoted by $(\mathbf{Q}, \mathbf{P}) \circ \mathbf{M}$, then specifies the following search rules for the seller. The first is a compound selection rule, which consists of selecting a family of bidder samples, $\mathbf{M}^T$, and the order of the bidder samples that the seller will search in each period, which is denoted by $\mathbf{M}$. The second is a stopping rule, according to which the seller decides whether or not stop searching. If the seller decides to stop searching in period t, then the allocation and payment will be implemented according to the stage mechanism of $(\mathbf{Q}_t, \mathbf{P}_t)$.

We assume that the seller announces the compound search mechanism of $(\mathbf{Q}, \mathbf{P}) \circ \mathbf{M}$ in period 0, and then fully commits to it thereafter. Under this full commitment assumption, we can restrict our attention to a direct mechanism. A well-known result in mechanism design (Myerson, 1981) is that, when the search mechanism is *incentive feasible*, the expected product revenue a seller can obtain from a truthful bidder i is just equal to that bidder's virtual value of $\psi(v_i)$.

C. Optimal Search Mechanism

If we replace a bidder's value of the product, v_i, by his virtual value of $\psi(v_i)$, then we can reformulate the seller's sequential search problem as Pandora's problem *a la* Weitzman (1979).[5] Specifically, for a given family of bidder samples, $\mathbf{M}^T$, we can think

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[5]Our model is distinct from his in at least two perspectives. First, we study a sequential search auction, where the targets for search (bidders) behave strategically, while, in Weitzman (1979), the targets for search are a set of boxes that behave non-strategically. Second, we study a case of sequential-and-simultaneous search procedure, while, in Weitzman (1979) and Crémer, Spiegel and Zheng (2007), they just study one-by-one sequential search procedures. Our model explicitly incorporates the seller's sampling strategy into consideration, and it takes the one-by-one sequential search models as special cases.

of the set M^j of bidders as an aggregated bidder j, who is characterized by the gross search cost of $c_{M^j} = \sum_{i \in M^j} c_i$, and the highest virtual value of the M^j bidders, which is denoted by

$$\psi\left(V_{M^j}^{(1)}\right) = \psi\left(\max_{i \in M^j} \{V_i\}\right), \tag{2}$$

where $V_{M^j}^{(k)}$ denotes the k-th highest order statistics of the M^j bidders' values, with distribution $F_{M^j}^{(k)}(v)$. To ease notation, we denote m^j as the number of bidders in the sample of M^j, and therefore $F_{M^j}^{(1)}(v) = F^{m^j}(x)$. We know that, when the search mechanism is incentive feasible, the maximum revenue a seller can obtain from the M^j bidders is just equal to the highest virtual value of them, that is, $\psi\left(V_{M^j}^{(1)}\right)$.

The seller needs to determine the optimal search procedure, which involves both a selection rule and a stopping rule.[6] We can formulate it as a dynamic programming (DP) problem for the seller. Supposing that, at the end of period t, the seller has invited N_t bidders and the highest bidder value is v, it then implies that the current best offer for the product is $\psi(v)$, given that the mechanism is incentive feasible. Taking the current highest value v as a state variable, the seller then faces the decision between stopping and continuing searching. If she decides to stop searching, then she will keep the offer of $\psi(v)$, and the product allocation and the bidder payments are implemented according to the rules of the stage mechanism. If she decides to continue searching, then she needs to decide which sample of bidders to contact in the next period.

We denote $J_t(v)$ as the value of having an offer of $\psi(v)$ at the end of period t, which the seller can either accept or reject. It is obvious that $J_{T+1}(v) = 0$ and $J_T(v) = \max\{\psi(v), 0\}$. For $t < T$, the Bellman equation for this sequential search problem with long-lived bidders is thus

$$J_t(v) = \max_{M_{t+1} \in N_t^c} \left\{ \psi(v), -c_{M_{t+1}} + \mathbb{E}J_{t+1}\left[\max\left\{v, X_{M_{t+1}}^{(1)}\right\}\right]\right\}, \tag{3}$$

where $\psi(v) \geq 0$ is the fall-back payoff of stop searching, and the other term in the curly braces is the maximum expected payoff of continuing searching.[7] It is easy to show that there exists a unique cutoff value, denoted by ξ_t^*, at which the seller is indifferent between stopping and continuing searching at the end of period t. Moreover, the value of ξ_t^* is solely determined by the characteristics of the bidder sample of M_{t+1}, that the

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[6]Given that a seller is fully committed to the search mechanism of $(\xi, \mathbf{M})$, the sequential search problem for the seller is analogous to the Pandora's problem in Weitzman (1979). For instance, when bidders bid truthfully, we may think of a set of M_t bidders as a single 'box' in Pandora's problem, except that the prize of the 'box' is now the highest virtual valuation, and the distribution of the prize is now the distribution of the highest valuation of the bidders. A well-known result is that, in an optimal sequential search procedure, the valuation cutoff ξ_t is necessarily decreasing over time (Weitzman, 1979).

[7]The assumption of $\psi(v) \geq 0$ is without loss of generality, as the seller can always stop searching and receive a zero revenue. In another word, without loss of generality, we can assume the current best off $v \geq r^*$.

seller will search in the right next period.

According to Weitzman (1979), the optimal stopping rule is as follows: if the current best offer $\psi(v) \geq \psi(\xi_t^*)$, then the seller stops searching, otherwise she will continue searching in the next period. The optimal selection rule is as follows: if a sample of M_{t+1} bidders is to be invited in the next period of $t+1$, then it must be the sample with the highest cutoff value among all the remaining uninvited bidder samples. It then implies that, in an optimal search procedure, the cutoff values of bidder samples, denoted as $\{\xi_t^*\}_{0 \leq t \leq T-1}$, must be declining over time.

The above optimal search rule also implies that, when $v = \xi_t^*$, the seller is indifferent between keeping the current best offer of $\psi(\xi_t^*)$, and continuing searching the sample M_{t+1} of bidders in the next period and then stopping right away. This is because, by searching for one more period, the maximum bidder value $\max\left\{v, X_{M_{t+1}}^{(1)}\right\} \geq \xi_t^* \geq \xi_{t+1}^*$, and thus, at the end of the next period, the seller will certainly prefer stopping to continuing searching. From the Bellman equation of (3), it then implies

$$(4) \qquad \psi(\xi_t^*) = \int_0^1 \max\left\{\psi(\xi_t^*), \psi(x)\right\} dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}},$$

where $F_{M_{t+1}}^{(1)}(x)$ is the distribution function of the highest value of the M_{t+1} bidders. The LHS of (4) is the current best offer of $\psi(\xi_t^*)$, and the RHS is the expected revenue net of search cost if continuing searching the sample M_{t+1} of bidders in period $t+1$. When these two values are equal, the seller is then indifferent between stopping and continuing searching for just another period.

It is easy to show that there is a unique solution to equation (4), we can define the solution as the optimal cutoff value, denoted by $\xi_t^* = \xi^*(M_{t+1})$. The value of $\xi^*(M)$ represents the seller's reservation value of contacting the set M of bidders, and it is determined by the gross search cost, c_M, and the distribution of the highest value, $F_M^{(1)}(x)$. For a given family of bidder samples, $\mathbf{M}^T$, the optimal search rule is then given as follows:

- Calculate the optimal cutoff value, $\xi^*\left(M^j\right)$, for each bidder sample $M^j \in \mathbf{M}^T$;

- Selection rule: If a bidder sample is to be contacted in period t, it must be the sample with the highest cutoff value among N_{t-1}^c. The optimal sampling rule $\mathbf{M} = (M_1, M_2, \cdots, M_T)$ is such that $\xi^*(M_1) \geq \xi^*(M_2) \geq \cdots \geq \xi^*(M_T)$.

- Stopping rule: If the current best offer of $\psi(v)$ is greater than the virtual value of the highest cutoff value of the remaining uninvited samples, then stop; otherwise, continue to search in the next period.

III.   Sequential Search Auction

In this section, we will show that the outcomes of the above optimal search mechanism are implementable through a sequence of second price auctions, with properly set reserve prices. Specifically, we propose the following rule of the sequential auction:

- In period 1, the seller invites a set $M_1 \in N_0^c$ of bidders to the auction at the search cost of $c_{M_1} = \sum_{i \in M_1} c_i$, and runs a second-price auction among the set $N_1 \equiv M_1$ of bidders, with a reserve price r_1;

- If an effective bid is submitted by a bidder $i \in N_1$ in period 1, then the game ends, and the payment and allocation are implemented according to the auction rule. If no effective bid submitted, the seller continues her searching by inviting a set $M_2 \in N_1^c$ of bidders in period 2, at the search cost of c_{M_2}. She then runs an auction among the accumulated set $N_2 \equiv M_1 \cup M_2$ of bidders, with a new reserve price r_2;

- The seller continues with this process, until the end of period T. For example, in any period $t < T$, if there is no effective bid submitted, then the seller continues her searching in period $t+1$, by inviting a set $M_{t+1} \in N_t^c$ of bidders at the search cost of $c_{M_{t+1}}$, and runs an auction among the accumulated set $N_{t+1} \equiv N_t \cup M_{t+1}$ of bidders, with a updated reserve price of r_{t+1}.

This sequential search auction is characterized by two components: a sequence of reserve prices, $\mathbf{r} \equiv \{r_t\}_{1 \leq t \leq T}$, and a sequence of bidder samples, $\mathbf{M} \equiv \{M_t\}_{1 \leq t \leq T}$. Second, as bidders are long-lived, once a bidder is invited in a period, he will stay in all the stage auctions from that period on, until the end of period T. Therefore, a bidder can always contribute to the auction revenue in all the remaining stage auctions. The assumption of long-lived bidders implies that the sequential search auction is analogous to a sequential search process with *full* recall.

The full commitment assumption implies that the seller announces the sequential auction of $(\mathbf{r}, \mathbf{M})$ in period 0, and fully commits to it afterwards. We will focus on the format of sealed-bid second price auction, which is strategically equivalent to a standard ascending open-auction. Under the full commitment assumption, if the sequential auction is incentive compatible, then bidding true value is a weakly dominant strategy for a bidder, if he intends to bid at all.[8]

A. Incentive Compatibility

We first investigate the incentive compatible conditions for bidders, and will study equilibria in the form of cutoff strategies. A cutoff strategy for a bidder is characterized by a vector of cutoff values, $\xi \equiv \{\xi_t\}_{1 \leq t \leq T}$, such that a bidder will bid his true value v in period t if $v \geq \xi_t$, and wait otherwise. Given that bidders are *ex-ante* homogeneous according to their value distributions, we will focus on symmetric equilibria. A symmetric cutoff equilibrium is an equilibrium where all the bidders adopt the same cutoff strategy in equilibrium.

Let $\bar{U}_t(v)$ be the value function of the expected payoff of a bidder with value v in period t, and $U_t^b(v)$ be the expected payoff of the bidder by offering an effective bid in

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[8]As the seller fully commits to a mechanism, without loss of generality, we assume that if $M_t = \emptyset$, then for all $t' \geq t$, $M_{t'} = \emptyset$ and $r_{t'} = r_{t-1}$. And we define $T = \max\{t \mid M_t \neq \emptyset\}$ for given $\mathbf{M}$. We also assume that the product must be sold with a positive probability in each period $t \leq T$; otherwise, one can construct an equivalent mechanism by replacing M_t by $M_t \cup M_{t+1}$ and $r_t = r_{t+1}$.

that period. It is then obvious that $\bar{U}_{T+1}(v) = 0$, and

$$U_t^b(v) = F_{N_t\setminus\{i\}}^{(1)}(\xi_t)(v - r_t) + \mathbb{I}_{\{v \geq \xi_t\}} \int_{\xi_t}^{v} (v - x) \, dF_{N_t\setminus\{i\}}^{(1)}(x).$$

where $\mathbb{I}$ is an indicator function, and $F_{N_t\setminus\{i\}}^{(1)}$ is the distribution function of the highest value of the set $N_t \setminus \{i\}$ of bidders. It is clear that

$$F_{N_t\setminus\{i\}}^{(1)}(v) = \begin{cases} F^{n_t}(v) & \text{if } i \notin N_t, \\ F^{n_t - 1}(v) & \text{if } i \in N_t, \end{cases}$$

where n_t denotes the number of bidders in N_t. The standard envelope theorem yields

$$\bar{U}_t(v) = \max\{U_t^b(v), \bar{U}_{t+1}(v)\},$$

which is non-decreasing, convex, and right-hand differentiable for all $v \in (0, 1]$.[9] It also implies that the optimal strategies of bidders are necessarily in the form of cut-off strategies, and therefore our previous assumption of cutoff strategies is without loss of generality. The following result is a direct application of envelope theorem, which shows that there exists a one-to-one mapping between the sequence of reserve prices, **r**, and that of the cutoff values, ξ.

LEMMA 1: *Given* $(\mathbf{r}, \mathbf{M})$*, in each period* $t \leq T$*, there exists a unique* ξ_t *such that each bidder* $i \in N_t$ *bids if and only if his true value* v *is greater than or equals to* ξ_t*. Furthermore, the expected payoff of a bidder with value* v *in period* t *is:*

$$(5) \qquad \bar{U}_t(v) = \begin{cases} \bar{U}_{t+1}(v) & \text{if } v < \xi_t, \\ \bar{U}_{t+1}(\xi_t) + \int_{\xi_t}^{v} F_{N_t\setminus\{i\}}^{(1)}(x) \, dx & \text{if } v \geq \xi_t. \end{cases}$$

From Lemma 1, the sequential auction of $(\mathbf{r}, \mathbf{M})$ can be equivalently represented by $(\xi, \mathbf{M})$, and hereinafter we think of $(\xi, \mathbf{M})$ as the sequential search auction instead. We already know that, in an optimal sequential search mechanism, the cutoff values are declining over time. When the cutoff values are declining over time, we have the following clear condition for a bidder's equilibrium cutoff strategy.

LEMMA 2 (cutoff condition): *For a given mechanism* $(\xi, \mathbf{M})$ *with declining cutoff values,* ξ_t *is uniquely determined by*

$$(6) \qquad F_{N_t\setminus\{i\}}^{(1)}(\xi_t)(\xi_t - r_t) = F_{N_{t+1}\setminus\{i\}}^{(1)}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F_{N_{t+1}\setminus\{i\}}^{(1)}(x) dx,$$

for $t < T$*, and* $\xi_T = r_T$*. Moreover, the reserve prices* $\{r_t\}_{1 \leq t \leq T}$ *is also decreasing in* t*.*

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[9]The derivation of the cut-off strategy is standard. It also appears in the literature of buy-price auction (Reynolds and Wooders, 2009; Chen et al., 2016) and sequential auctions with information acquisition costs (Crémer, Spiegel and Zheng, 2009). Here we apply envelope theorem.

Since $i \in N_t$ implies $i \in N_{t+1}$, the other equivalent expression of (6) is

$$(7) \qquad F_{N_t-1}^{(1)}(\xi_t)(\xi_t - r_t) = F_{N_{t+1}-1}^{(1)}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F_{N_{t+1}-1}^{(1)}(x)dx$$

for $t < T$, and $\xi_T = r_T$ for $t = T$. Here, by an abuse of notation, $N_t - 1$ represents a set constructed by removing an arbitrary bidder from N_t, and thus $F_{N_t-1}^{(1)}(v) = F^{n_t-1}(v)$.

B.   *Optimal Cutoff Values*

The expected auction profit for the seller is equal to the expected auction revenue minus the expected gross search cost. For a given mechanism $(\xi, \mathbf{M})$ with declining cutoff values, the expected auction profit can be represented by

$$(8) \qquad \pi(\xi, \mathbf{M}) = \sum_{t=1}^{T} F_{N_t-1}^{(1)}(\xi_{t-1})\left[R_t(N_t) - c_{M_t}\right],$$

where $F_{N_0}^{(1)}(\xi_0) \equiv 1$ and $R_t(N_t)$ is the expected revenue of the stage auction in period t, conditional on it happens. It is worthy of attention that, in period t, there are M_t strong bidders and N_{t-1} weak bidders in the auction. In another word, the values of the M_t new bidders are independent draws from F on $[0, 1]$, while those of the N_{t-1} weak bidders are independent draws from the truncated distribution of $F(v|\xi_{t-1}) = \Pr(V \leq v | V \leq \xi_{t-1})$. Note that, if a distribution function F is of *IFR*, then its truncated distribution function $F(\cdot|\xi_{t-1})$ is also of *IFR*.

Substituting the bidders' equilibrium cutoff strategies of (6) into (8), we get the following expression of the expected auction profit.

LEMMA 3:   *For a given mechanism $(\xi, \mathbf{M})$ with declining cutoff values, the expected auction profit is*

$$(9) \quad \pi(\xi, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} \psi(x)\, dF_{N_t}^{(1)}(x) + \sum_{t=1}^{T} F_{N_t-1}^{(1)}(\xi_{t-1})\left[\int_{\xi_{t-1}}^{1} \psi(x)\, dF_{M_t}^{(1)}(x) - c_{M_t}\right],$$

where $\xi_0 \equiv 1$ and $F_{N_0}^{(1)}(\cdot) \equiv 1$.

Another more intuitive expression of (9) is that

$$(10) \qquad \pi(\xi, \mathbf{M}) = \sum_{t=1}^{T} F_{N_t-1}^{(1)}(\xi_{t-1})\left[\int_{\xi_t}^{1} \psi(x)\, dG_{N_t}^{(1)}(x) - c_{M_t}\right],$$

where $G_{N_t}^{(1)}(x)$ is the distribution of the highest value of the N_t bidders. Remember that, among the N_t bidders, N_{t-1} bidders' values are independent draws from the truncated

distribution of $F(v|\xi_{t-1})$. It then follows that

$$G_{N_t}^{(1)}(x) = \begin{cases} F_{N_t}^{(1)}(x)/F_{N_{t-1}}^{(1)}(\xi_{t-1}), & \text{if } x \in [0, \xi_{t-1}]; \\ F_{M_t}^{(1)}(x), & \text{if } x \in (\xi_{t-1}, 1]. \end{cases}$$

The expression of also demonstrates the standard result of revenue equivalence theorem. That is, the expected revenue of a stage auction is just equal to the highest virtual value of the N_t bidders, given that it is greater than the cutoff value of ξ_t.

The expression of (9), as well as (10), also demonstrates the insight of Myerson (1981), that the expected revenue of a stage auction is just equal to the highest virtual value of the N_t bidders, given that it is greater than the cutoff value of ξ_t. Yet, different from the standard results for static auctions with symmetric bidders, (9) provides a nice formula for a sequential search auction with asymmetric bidders. In our model, as bidders are *ex-ante* homogeneous, their virtual value functions are the same, e.g. $\psi(v)$. The stage auction in period t happens with probability $F_{N_{t-1}}^{(1)}(\xi_{t-1})$, where there are M_t strong bidders, and N_{t-1} weak bidders whose values are truncated above from ξ_{t-1}. When the highest value of the bidders is between ξ_t and ξ_{t-1}, all the N_t bidders are competing with each other to win the product. This corresponds to the part of $\int_{\xi_t}^{\xi_{t-1}} \psi(x) dF_{N_t}^{(1)}(x)$ in (9). When the highest value is greater than ξ_{t-1}, then the N_{t-1} weak bidders are strictly dominated by the M_t strong bidders, and the competition happens only among the M_t strong bidders. This corresponds to the other term of $\int_{\xi_{t-1}}^{1} \psi(x) dF_{M_t}^{(1)}(x)$ in (9).

PROPOSITION 1 (optimal cutoffs): *For a given mechanism $(\xi, \mathbf{M})$ with declining cutoff values, the expected auction profit, $\pi(\xi, \mathbf{M})$, is quasi-concave in ξ_t. The sequence of optimal cutoff values, $\{\xi_t^*\}_{1 \leq t \leq T}$, is the unique solution to*

$$(11) \qquad c_{M_{t+1}} = \int_{\xi_t^*}^{1} [\psi(x) - \psi(\xi_t^*)] dF_{M_{t+1}}^{(1)}(x), \text{ for } 1 \leq t < T,$$

and $\psi(\xi_T^*) = 0$ for $t = T$.

PROOF:

For $t < T$, due to Lemma 3, the derivative of $\pi(\xi, \mathbf{M})$ with respect to ξ_t is

$$\begin{aligned} \frac{\partial \pi}{\partial \xi_t} &= \psi(\xi_t) \left[f_{N_{t+1}}^{(1)}(\xi_t) - f_{N_t}^{(1)}(\xi_t) \right] \\ &\quad + f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^{1} \psi(x) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right] - \psi(\xi_t) F_{N_t}^{(1)}(\xi_t) f_{M_{t+1}}^{(1)}(\xi_t) \\ &= f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^{1} (\psi(x) - \psi(\xi_t)) dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right] = f_{N_t}^{(1)}(\xi_t) \cdot \eta(\xi_t) = 0. \end{aligned}$$

Note that

$$\frac{\partial \eta}{\partial \xi_t} = -\psi'(\xi_t) \int_{\xi_t}^{1} dF_{M_{t+1}}^{(1)}(x) < 0,$$

and then $\frac{\partial \pi}{\partial \xi_t}$ changes its sign from positive to negative at most once. $\pi(\xi, \mathbf{M})$ is then quasi-concave in ξ_t, and the first order condition is also sufficient. When $t = T$, $\frac{\partial \pi}{\partial \xi_T} = -\psi(\xi_T) f_{N_T}^{(1)}(\xi_T)$. It is obvious that π is concave in ξ_T given the IFR assumption, and thus $\psi(\xi_T^*) = 0$. ∎

The optimal cutoff value of (11) also reflects the condition for optimal stopping rule in sequential search, as shown in (4). For instance, to continue searching the sample M_{t+1} of bidders, the seller needs to pay a gross search cost of $c_{M_{t+1}}$, which is the LHS of (11). On the other hand, given the current best offer of $\psi(\xi_t^*)$, the RHS of (11) is the increment in expected auction revenue, which is decreasing in ξ_t^*. When the two parts are equal, the seller is indifferent between keeping the offer of $\psi(\xi_t^*)$ and continuing searching the sample M_{t+1} of bidders in the next period.

Moreover, the optimal cutoff value of ξ_t^* demonstrates a one-step-ahead property, in the sense that it just depends on the sample of bidders who are invited in the next period of $t + 1$, not further. This property is a direct implication of the facts that the cutoff values are decreasing over time under optimal search, and that bidders are assumed to be long-lived. In another word, at the end of period t, if the seller is indifferent between stopping and continuing searching, she will (weakly) prefer stopping in the next period of $t + 1$, because her fall-back revenue increases while the optimal cutoff value in the next period of $t + 1$ decreases.

Our formula of (11) generalizes the famous result in infinite sequential search problems (Weitzman, 1979). In these problems, Pandora inspects a number of closed boxes, each with a random prize distributed according to F_i on $[0, 1]$; to open a box i, she needs to pay a search cost of c_i. Pandora inspects boxes one-by-one, and her objective is to maximize the expected prize discovered, net of the gross search cost. Weitzman shows that the optimal search procedure involves allocating a unique reservation prize ξ_i^* for each box i, which is the unique solution to

$$
(12) \qquad c_i = \int_{\xi_i}^{1} (x - \xi_i) \, dF_i(x).
$$

Supposing Pandora already has a fall-back prize of ξ_i, then the RHS of (12) is the increment in expected utility if she inspects box i, and the LHS is the search cost. Therefore, if the fall-back prize ξ_i is equal to the reservation prize ξ_i^*, Pandora is indifferent between stopping and continuing inspecting box i.

Our formula of (11) generalizes Weitzman's result of (12) in two important dimensions. First, in our model, the target for search are strategic bidders, rather than non-strategic boxes as in Pandora's problem. This extension is important, as it well fits many important situations in markets, such as M&As, job recruitment, matching in marriage markets, and so on. Second, we extend Weitzman's result from one-by-one sequential search to the more general case of compound search, where a seller searches both sequentially and simultaneously. As a result, we could think of Weitzman's formula of (12) as a special case of our result of (11).

To provide further characterization of the optimal search procedure, it is helpful to

define the optimal cutoff value for a sample M of bidders by $\xi^*(M)$. Specifically, for any $M \subseteq N$ with $M \neq \emptyset$, $\xi^*(M)$ is the unique solution to the equation of $c_M = \int_\xi^1 [\psi(x) - \psi(\xi)] dF_M^{(1)}(x)$. Or equivalently,

$$(13) \qquad c_M = \int_{\xi^*(M)}^1 \left[1 - F_M^{(1)}(x)\right] d\psi(x).$$

From the definition, $\xi^*(M)$ measures the seller's reservation value of searching the sample M of bidders, for the purpose of profit maximization. For instance, if the seller's current best offer of $\psi(\xi)$ is greater than $\psi(\xi^*(M))$, she will have no incentives to further search the sample M of bidders. The following corollary provides some intuitive characterizations of $\xi^*(M)$.

COROLLARY 1: *For two bidder samples $M, M' \subseteq N$,*

1) *if the cardinality $|M| = |M'|$, then*

$$c_M < c_{M'} \implies \xi^*(M) > \xi^*(M');$$

2) *if $c_M = c_{M'}$, then*

$$|M| < |M'| \implies \xi^*(M) < \xi^*(M').$$

The above corollary states that, for two samples of bidders, if they have the same sample size (e.g. the number of bidders), then the sample with smaller gross search cost is more valuable for the seller. Second, if their gross search costs are the same, then the sample with more bidders is more valuable for the seller.

Besides the search cost and the sample size, we may also examine how the optimal cutoff value changes with the value distribution of F. We will show that when the value distribution becomes more riskier, in terms of mean preserving spread (MPS), the optimal cutoff value of a bidder sample increases. To be specific, let $\{F(\cdot, \sigma)\}$ be a family of distributions indexed by σ, with common support on $[0, 1]$. For $\sigma' > \sigma$, we say $F(\cdot, \sigma')$ is more risky than $F(\cdot, \sigma)$ in terms of MPS, if $\int_0^v [F(x, \sigma') - F(x, \sigma)] dx \geq 0$ for all $v \in [0, 1]$, and with strict equality when $v = 1$. Correspondingly, for a sample M of bidders with value distribution $F(x, \sigma)$, we denote its optimal cutoff value by $\xi^*(M; \sigma)$, and will show that $\xi^*(M; \sigma)$ increases in σ, the index of riskiness of value distribution.

COROLLARY 2: *For a sample M of bidders, if their values are independent draws from two distributions of $F(\cdot; \sigma)$ and $F(\cdot; \sigma')$ respectively, such that $F(\cdot; \sigma') \succ_{MPS} F(\cdot; \sigma)$ in terms of MPS, then the optimal cutoff value*

$$\xi^*(M; \sigma') > \xi^*(M; \sigma).$$

In the remaining part of this section, we will derive a succinct formula for the maximum expected profit when the reservation values are set at the optimal levels. We will also show that our formula for maximum profit can take the result in stationary and infinite horizon search problems as a special case.

First, comparing (11) with (13), it is clear that the optimal cutoff value of $\xi_t^* = \xi^*(M_{t+1})$ for $1 \leq t < T$. This result coincides with the optimal cutoff condition of (4). It then implies that a sequential search auctions can implement the outcomes of an optimal search mechanism. Specifically, for a given family of bidder samples, $\mathbf{M}^T$, the optimal sampling rule is a permutation of it such that $\xi^*(M_t) \geq \xi^*(M_{t+1})$ for $1 \leq t < T$. With an abuse of notation, we denote this optimal sampling rule for $\mathbf{M}^T$ also by $\mathbf{M} = \{M_t\}_{1 \leq t \leq T}$. Then for a given family of bidder samples of $\mathbf{M}^T$, the sequential search auction of $(\xi^*, \mathbf{M})$ implements the outcomes of an optimal search mechanism, where $\xi^* = \{\xi^*(M_{t+1})\}_{1 \leq t \leq T}$ with $\xi^*(M_{T+1}) \equiv r^*$.

Second, substituting the optimal cutoff values of (11) into the expected auction profit of (9), we then have the following expression of maximum expected auction profit.

LEMMA 4: *Given a family of bidder samples $\mathbf{M}^T$, the maximum expected auction profit is*

$$(14) \qquad \pi^*\left(\mathbf{M}^T\right) = \pi\left(\xi^*, \mathbf{M}\right) = \sum_{t=1}^{T} \int_{\xi^*(M_{t+1})}^{\xi^*(M_t)} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x),$$

where $\xi^*(M_{T+1}) \equiv r^*$.

A well-known result in stationary and infinite-horizon (SIH) search problem is that, when following the optimal search procedure, the maximum expected search profit is equal to the value of the optimal reservation value $\psi(\xi^*)$ that solves $c = \int_{\xi^*}^{1} [1 - F(x)] d\psi(x)$ (Lippman and McCall, 1976). If we denote the optimal search profit of a SIH search problem by $\pi^{SIH}(\xi^*) = \psi(\xi^*)$, our result of (18) incorporates this result of SIH sequential search as a special case.

For the purpose of comparison, let us consider the case of homogeneous bidders ($c_i = c$) and one-by-one sequential search ($|M_t| = 1$). Under these assumptions, when $T \to \infty$, our sequential search problem converges to a SIH search problem *a la* Lippman and McCall (1976). Applying the formula of (14) and taking the limit $T \to \infty$, the maximum expected profit by following an optimal search strategy is thus

$$(15) \qquad \lim_{T \to \infty} \pi^*(\mathbf{M}) = \lim_{T \to \infty} \int_{r^*}^{\xi^*} \left[1 - F^T(x)\right] d\psi(x) = \psi(\xi^*) = \pi^{SIH}(\xi^*).$$

When invitation costs are different over the bidders, the seller may invite more bidders in earlier stages.

EXAMPLE 1 (2-period with 3-buyers): *Consider $F = x$ and $c_1 = 0 \leq c_2 \leq c_3 = \frac{1}{16}$. By (13), the optimal cutoff $\xi^*(\{i\})$ for inviting a single bidder i is $\xi^*(\{i\}) = 1 - \sqrt{c_i}$ and the optimal cutoff $\xi^*(\{i, j\})$ for inviting two bidders $\{i, j\}$ is the solution to $c_i + c_j = \frac{2}{3}(1 - \xi)^2(2 + \xi)$. The cutoffs are plotted in Figure 1. As the cutoffs for inviting two bidders are all greater than $r^* = 1/2$, consider the three sampling partitions, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, and $\{\{3\}, \{1, 2\}\}$. For each sampling partition, invite the part with a higher cutoff in the first period and invite the other part in the second period. As $\xi^*(\{1\}) > \xi^*(\{2, 3\})$ and $\xi^*(\{3\}) > \xi^*(\{1, 2\})$, the candidate sampling rule*
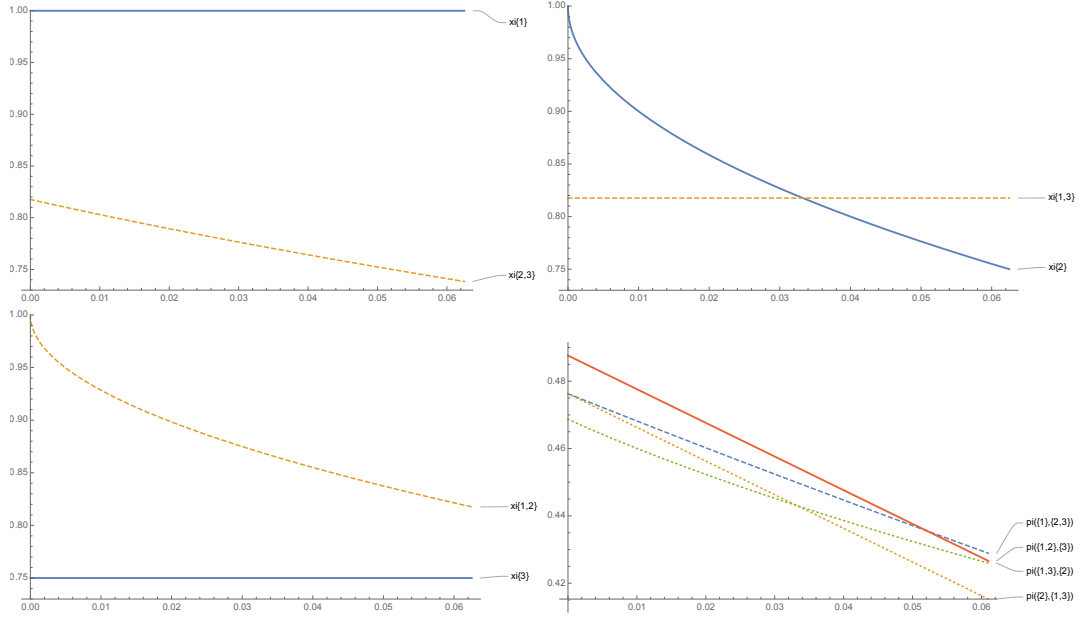
FIGURE 1. THREE PARTITIONS AND FOUR SAMPLING STRATEGIES OVER TWO PERIODS

Note: See Example 1. Consider three bidders with heterogeneous costs, $c_1 = 0 \leq c_2 \leq c_3 = 1/16$. As $c_i \leq 1/16$, all the bidders are supposed to be invited and three sampling partitions are available: $\{\{1\}, \{2,3\}\}$, $\{\{2\}, \{1,3\}\}$, and $\{\{3\}, \{1,2\}\}$. For each partition, the corresponding optimal cutoffs are shown in each subfigure, where c_2 changes along the horizontal axis. Each of the sampling partition $\{\{1\}, \{2,3\}\}$ and $\{\{3\}, \{1,2\}\}$ yields the sampling order $(\{1\}, \{2,3\})$ and $(\{1,2\}, \{3\})$ as the order of the cutoffs is clear in each partition. For the sampling partition $\{\{2\}, \{1,3\}\}$, $(\{2\}, \{1,3\})$ should be considered as a sampling rule if $c_2 < \bar{c} \approx 0.03327$, and $(\{1,3\}, \{2\})$ if $c_2 > \bar{c}$. The bottom-right subfigure depicts the expected profits of the four sampling strategies. For $c_2 < \bar{c}^* \approx 0.05208$, inviting $\{1,2\}$ in the first period and $\{3\}$ in the second period is optimal. If $c > \bar{c}^*$, on the other hand, inviting only $\{1\}$ in the first period and inviting remaining $\{2,3\}$ in the second period is optimal.

is $(\{1\}, \{2,3\})$ and $(\{1,2\}, \{3\})$, respectively associated with the sampling partitions $\{\{1\}, \{2,3\}\}$ and $\{\{3\}, \{1,2\}\}$. Due to Lemma 4, the profit for each sampling rule can be computed as follows:

$$(16) \qquad \pi^*(\{1\}, \{2,3\}) = \int_{\xi^*(\{1\})}^{\xi^*(\{2,3\})} (1 - F(x)) d\psi(x) + \int_{r^*}^{\xi^*(\{2,3\})} (1 - F^3(x)) d\psi(x)$$

$$(17) \qquad \pi^*(\{1,2\}, \{3\}) = \int_{\xi^*(\{1,2\})}^{\xi^*(\{3\})} (1 - F^2(x)) d\psi(x) + \int_{r^*}^{\xi^*(\{3\})} (1 - F^3(x)) d\psi(x)$$

Another candidate from the partition $\{\{2\}, \{1,3\}\}$ depends on c_2. If $c_2 < \bar{c} \approx 0.03327$, then $\xi^*(\{2\}) > \xi^*(\{1,3\})$ and the profit from the sampling rule $(\{2\}, \{1,3\})$ is greater than that from $(\{1,3\}, \{2\})$; otherwise, $(\{1,3\}, \{2\})$ is better than the other. However, one can confirm that any of these sampling rules is dominated by either $(\{1\}, \{2,3\})$ or

($\{1,2\},\{3\}$) *as Figure 1 illustrates. Comparing (16) and (17), the optimal sampling rule* $\mathbf{M}^*$ *is* ($\{1,2\},\{3\}$) *for* $c_2 < \bar{c}^*$ *and* ($\{1\},\{2,3\}$) *otherwise, where* $\bar{c}^* \approx 0.05208$. *That is, the invitation cost of bidder 2 is lower than a certain threshold, inviting him together with bidder 1 is better; however, if the cost is greater than the threshold, the seller is better to invite him in the second period.*

C.   Optimal Sampling Rule

Now we turn to the other part of an optimal search mechanism – the optimal sampling rule. We have known that, for a given family of bidder samples, $\mathbf{M}^T$, the optimal sampling rule of $\mathbf{M}$ is such that $\xi^*(M_t)$ is decreasing in t. In this section, we will provide more characterizations of the optimal choice of $\mathbf{M}^T$ and the sampling rule of $\mathbf{M}$, and our focus is on optimal search intensity in each period.

To fix the idea of search intensity, we consider the case of homogeneous bidders, where all bidders have the same value distribution F, and the same unit search cost, i.e., $c_i = c$ for all $i \in N$. For ease of notation, we denote $m_t = |M_t|$ and $n_t = |N_t|$ as the cardinality of M_t and N_t respectively, and $\xi^*(m)$ as the optimal cutoff value for searching a sample M of bidders.

In this case of homogeneous bidders, a seller's sampling rule is simply represented by a sequence of sample sizes, denoted by $\mathbf{m} = (m_1, m_2, \cdots, m_T)$. Intuitively, the degree of search intensity in period t is simply measured by the sample size of m_t. We say a seller searches more intensively in period t' than in period t, if and only if $m_{t'} \geq m_t$. The following result shows that the optimal cutoff value of $\xi^*(m)$ is decreasing in m.

LEMMA 5:   *Suppose* $c_i = c$ *for all* $i \in N$. *The optimal cutoff value of* $\xi^*(m)$ *for inviting a sample M of bidders is strictly decreasing in its sample size of m. That is, for any two sets of bidders, $M, M' \subseteq N$, if $m < m'$, then*

$$\xi^*(m) > \xi^*(m').$$

PROOF:
For any $M \subseteq N$, due to (13), we have

$$c = \int_{\xi^*(m)}^{1} \frac{1}{m}(1 - F^m(x)) \, d\psi(x).$$

As $F(x) < 1$ for $x \in [0, 1)$, $(1 - F^m(x))/m$ is strictly decreasing in m. Therefore, when m increases, $\xi^*(m)$ must decreases, so as to keep the above equation to hold.

The result is more striking than it first looks. We know the optimal cutoff value $\xi^*(m)$ is decreasing in unit search cost c and increasing in sample size m. The above result then shows that, when bidders are homogeneous, the benefit of increasing competition by inviting one more bidder is strictly dominated by the unit cost of invitation. As a result, the maximum value of the optimal cutoff is achieved when the seller just contacts one bidder. Another implication is that, when a seller is not constrained by a finite deadline,

it is optimal for her to invite the bidders one-by-one and conduct pure sequential search, as it generates the highest expected profit, as shown in (15).

In the remaining parts of this section, we will show two results on optimal sampling rule for this problem of sequential search with a finite deadline. First, the optimal search intensity (sample size) is weakly increasing over time. Second, if we consider another interpretation of sample size and allow it to take real values, then constant search intensity is never optimal.

The first result of increasing search intensity is a direct implication of the above Lemma 9 and the fact that, in an optimal search mechanism, the optimal cutoff value ξ_t^* is necessarily decreasing in t. Therefore, it is optimal for the seller to sample increasingly more bidders in each period when the deadline approaches. If we denote $\mathbf{m}^* = (m_1^*, m_2^*, \cdots, m_T^*)$ as the optimal sampling rule for the seller, we have the following result.

PROPOSITION 2 (Optimal Sampling): *Suppose $c_i = c$ for all $i \in N$. The optimal search intensity is increasing in t, that is, for $1 \le t < T$,*

$$m_t^* \le m_{t+1}^*.$$

This is a second main result of this paper, which states that, in the presence of positive search cost and a finite deadline for completing a transaction, a seller will search increasingly more intensively when the deadline gets closer. In another word, in the first round of the transaction, she will contact the fewest number of buyers for participation. Each time when she does not receive a satisfactory offer for the product, in the next period, she will invite more buyers to join the competition.

This result may help our understanding of the puzzle that why many important selling processes seem non-competitive in markets, such as in M&As, sequential academic recruitment, matching in marriage markets, and so on. For instance, if we interpret M&As as a finite sequential search process for a seller, then it is optimal for her to contact relatively fewer bidders and set higher reserve prices in the early stages of the transaction. However, when the deadline gets closer and the seller has fewer opportunities to increase her payoff, it is better for her to lower the reserve prices and invite more bidders to the transaction. Our model may help explain why one-on-one negotiation can be the dominant selling mechanism in M&As.

Second, we will show that, when search intensity is continuous, i.e., m_t can take real values, the optimal search intensity must be strictly increasing in t. For continuous search intensity, there are many interpretations, i.e., it could be the periodical expenditure spent on job searching, or it could be the periodical effort invested in interviewing a job candidate. We already show that, under optimal search, the maximum expected auction profit is

$$(18) \qquad \pi^*(\mathbf{m}) = \sum_{t=1}^{T} \int_{\xi^*(m_{t+1})}^{\xi^*(m_t)} \left[1 - F_{n_t}^{(1)}(x) \right] d\psi(x)$$

Now we assume the degree of search intensity, measured by m_t, can take real values.

For a marginal increase of search intensity in period τ, the increase in expected profit is denoted by $D_\tau \pi^* (\mathbf{m}) = \partial \pi^* (\mathbf{m}) / \partial m_\tau$. Simple calculation from (18) gives that

$$
\begin{aligned}
D_\tau \pi^* (\mathbf{m}) \;=\; & F_{n_{\tau-1}}^{(1)} (\xi^* (m_\tau)) \left[-\int_{\xi^*(m_\tau)}^1 F_{m_\tau}^{(1)} (x) \ln F(x) \, d\psi(x) - c \right] \\
& - \sum_{t=\tau}^{T} \int_{\xi^*(m_{t+1})}^{\xi^*(m_t)} F_{n_t}^{(1)} (x) \ln F(x) \, d\psi(x).
\end{aligned}
$$

The marginal profit of $D_\tau \pi^* (\mathbf{m})$ can be decomposed into two parts. The first is the instant marginal profit of

$$
-F_{n_{\tau-1}}^{(1)} (\xi^* (m_\tau)) \left\{ \int_{\xi^*(m_\tau)}^1 F_{m_\tau}^{(1)} (x) \ln F(x) \, d\psi(x) + \int_{\xi^*(m_{\tau+1})}^{\xi^*(m_\tau)} G_{n_\tau}^{(1)} (x) \ln F(x) \, d\psi(x) + c \right\}
$$

where $F_{n_{\tau-1}}^{(1)} (\xi^* (m_\tau))$ is the probability for the stage auction in period τ to happen, and the term in the curly braces is the stage marginal profit of inviting one more bidder in that period. The second part is the continuation marginal profit of

$$
(19) \qquad\qquad - \sum_{t=\tau+1}^{T} \int_{\xi^*(m_{t+1})}^{\xi^*(m_t)} F_{n_t}^{(1)} (x) \ln F(x) \, d\psi(x),
$$

due to the effect that there is one more bidder participating in all the following stage-auctions. A necessary condition for a sampling rule of $\mathbf{m}$ to be optimal is that

$$
(20) \qquad\qquad D_1 \pi^* (\mathbf{m}) = D_2 \pi^* (\mathbf{m}) = \cdots = D_T \pi^* (\mathbf{m}) \geq 0.
$$

If the total number of bidders is sufficiently large and the resource constraint of $\sum_{t=1}^T m_t \leq n$ is not binding, then the $D_\tau \pi^*$'s are all equal to 0 in optimum. Otherwise, the marginal profit in all periods are equal and strictly positive in optimum.

The following result shows that, when search intensity is constant across all the periods, the marginal profit of increasing search intensity in a later period is strictly greater than that in an earlier period. In another word, the optimal search intensity must be strictly increasing over time, when the degree of search intensity can take real values. In other words, the equality in Proposition 2 may hold but only due to integer restrictions.

PROPOSITION 3 (Strict Monotonicity): *The optimal search intensity is strictly decreasing over time, when the search intensity can take real values.*

PROOF:

Suppose $m_\tau = m_{\tau'}$, then $\xi^* (m_\tau) = \xi^* (m_{\tau'})$ and

$$
D_{\tau'} \pi^* - D_\tau \pi^* = \left[F_{n_{\tau-1}}^{(1)} (\xi_\tau^*) - F_{n_\tau}^{(1)} (\xi_\tau^*) \right] \left[\int_{\xi^*(m_\tau)}^1 F_{m_\tau}^{(1)} (x) \ln F(x) \, d\psi(x) + c \right] > 0,
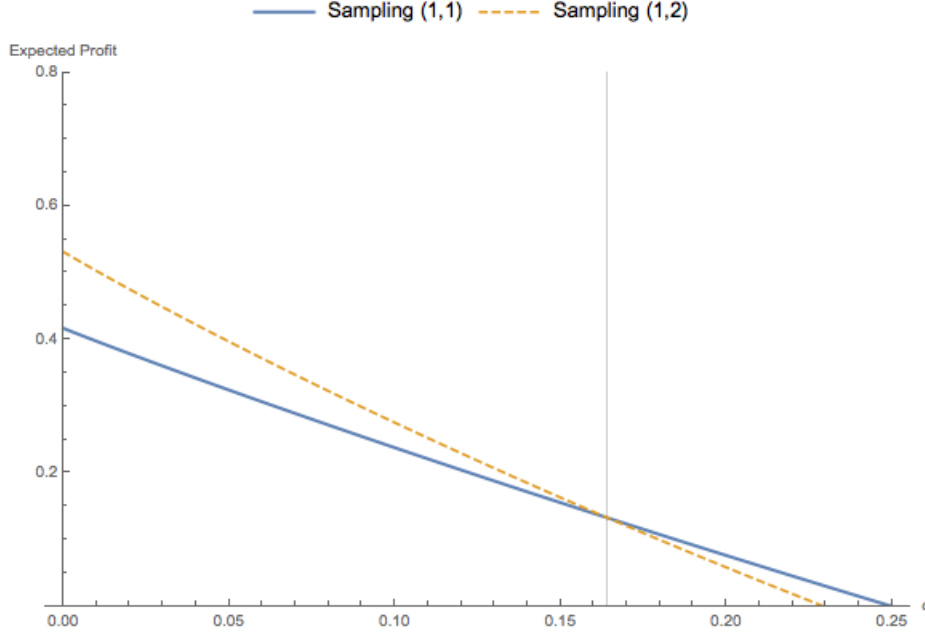$$

FIGURE 2. EXPECTED PROFIT IN THE 2-PERIOD 3-BIDDER PROBLEM

Note: See Example 2. When the invitation cost is lower than $c^* \approx 0.164$, it is optimal to invite one bidder in the first period and two the remaining bidders in the second period. If $c^* < c < 1/4$, however, it is optimal to invite just one bidder sequentially in both periods. For a higher invitation cost of $c > 1/4$, not inviting is optimal, as the expected profit from inviting any bidder is negative.

as $n_\tau = n_{\tau-1} + m_\tau$ and

$$\int_{\xi^*(m_\tau)}^1 F_{m_\tau}^{(1)}(x) \ln F(x) \, d\psi(x) + c = -\left[1 - F_{m_\tau}^{(1)}(\xi^*(m_\tau))\right] \frac{\partial \xi^*}{\partial m_\tau} > 0,$$

which contradicts to the necessary condition (20) of optimality. ∎

It is a well-known result in the literature that, when bidders (or boxes) are homogeneous and search is infinite, then the optimal reservation values (prices) are constant across all the periods. This is because it is a stationary and infinite horizon problem for a seller, and in each period she faces the same decision as in the previous period. However, when a seller faces a finite deadline, this dynamic programming problem becomes non-stationary. Our result shows that, for finite sequential search problems, if search intensity can take continuous values, then optimal search intensity is strictly increasing over time, and constant search intensity is never optimal.

The example below considers a simple 2-period search problem with 3-homogeneous long-lived bidders.

EXAMPLE 2: *Bidders are ex-ante homogeneous, with uniform value distribution on $[0,1]$ and unit search cost of $c \in (0, 1/4)$. In this case, the virtual value function is*

$\psi(x) = 2x - 1$. *We denote the sampling rule by* $\mathbf{m} = (m_1, m_2)$, *where m_t is the bidder sample size in period t. We know that, under optimal search procedure, it must be true that $m_1 \leq m_2$, and the candidates for optimal sampling rule is thus* $\mathbf{m} = (1,1)$ *or* $(1,2)$, *depending on the value of c. From formula (13), the optimal cutoff value for different sample size is given by:*

- *If $m = 1$, then $\xi^*(1)$ is the solution to $c = (1 - \xi)^2$, e.g., $\xi^*(1) = 1 - \sqrt{c}$;*

- *If $m = 2$, then $\xi^*(2)$ is the solution to $2c = \frac{2}{3}(1 - \xi)^2(2 + \xi)$.*

Case 1: *If $c = \frac{1}{16}$, then $\xi^*(1) = \frac{3}{4}$ and $\xi^*(2) \approx 0.738$. For a sampling rule of $(1,1)$, the expected profit, from (18), is*

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)] \, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) = \frac{29}{96} \approx 0.302.$$

The expected profit for a sampling rule of $(1,2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)] \, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.365.$$

Therefore, the sampling rule of $(1,2)$ is optimal.

Case 2: *If $c = \frac{5}{24}$, then $\xi^*(1) \approx 0.544$ and $\xi^*(2) = 0.5$. For a sampling rule of $(1,1)$, the expected profit, from (18), is*

$$\pi(1,1) = \int_{\xi^*(1)}^{\xi^*(1)} [1 - F(x)] \, d\psi(x) + \int_{r^*}^{\xi^*(1)} \left[1 - F^2(x)\right] d\psi(x) \approx 0.0634.$$

The expected profit for a sampling rule of $(1,2)$ is

$$\pi(1,2) = \int_{\xi^*(2)}^{\xi^*(1)} [1 - F(x)] \, d\psi(x) + \int_{r^*}^{\xi^*(2)} \left[1 - F^3(x)\right] d\psi(x) \approx 0.0417.$$

Therefore, the sampling rule of $(1,1)$ now is optimal.

IV. Efficient Mechanism

It is well-known that an optimal static auction usually leads to inefficient outcomes, due to the possibility of no trade in some states, and sometimes biased allocations. In our model of sequential search auctions, there is a third source of inefficiency in an optimal search mechanism, due to a seller's inefficient search procedures. First, the optimal search mechanism may exclude some bidders who would have been invited in an efficient search mechanism. Second, the optimal search mechanism may provide excessive incentives for a profit-maximizing seller to invite bidders (Li, 2017; Li and Xu,

2016). Third, the order of searching bidders in an optimal mechanism may be different from that in an efficient mechanism.

A.   Efficient Cutoffs

In an efficient mechanism, a seller's objective is to maximize expected social welfare. Following the same analysis as in Section II.C, we can show that an efficient search mechanism is also featured by declining cutoff values. In addition, although the seller's objective has changed from profit to welfare maximization, the incentive problem for bidders remains the same as in Section III.A. As a result, we still have the same results of (6) on bidders' equilibrium cutoff strategies. Given a mechanism $(\xi, \mathbf{M})$ with declining cutoff values, similar to (8), the expected social welfare is

$$(21) \qquad W(\xi, \mathbf{M}) = \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[W_t(N_t) - c_{M_t} \right],$$

where $W_t(N_t)$ is the expected social welfare of the stage auction in period t, conditional on it happens. The value of social welfare is equal to the value of the winner of the product. In a second price auction, as the bidder with the highest value wins the auction, the social welfare is just equal to the highest value of the participating bidders. As in Section IV, by substituting the cutoff condition of (6) into (21), we have the following result.

LEMMA 6: *For a given mechanism $(\xi, \mathbf{M})$ with declining cutoff values, the expected social welfare is*

$$(22) \qquad W(\xi, \mathbf{M}) = \sum_{t=1}^{T} \int_{\xi_t}^{\xi_{t-1}} x \, dF_{N_t}^{(1)}(x) + \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[\int_{\xi_{t-1}}^{1} x \, dF_{M_t}^{(1)}(x) - c_{M_t} \right],$$

where $\xi_0 \equiv 1$ and $F_{N_0}^{(1)}(\xi_0) \equiv 1$.

Similar to (10) in the case of expected auction profit, we could provide another more intuitive expression of the expected social welfare as follows

$$(23) \qquad W(\xi, \mathbf{M}) = \sum_{t=1}^{T} F_{N_{t-1}}^{(1)}(\xi_{t-1}) \left[\int_{\xi_t}^{1} x \, dG_{N_t}^{(1)}(x) - c_{M_t} \right],$$

where $G_{N_t}^{(1)}(x) = F_{N_{t-1}}^{(1)}(x | \xi_{t-1}) F_{M_t}^{(1)}(x)$ is the distribution of the highest value of the N_t bidders. Remember that among the N_t bidders, the set N_{t-1} of bidders are weak bidders, while the sample M_t of bidders are strong ones. Another measure of the social welfare is the sum of auction revenue, which is equal to the winning bidder's virtual value of $\psi(x)$, and the winning bidder's information rent, which is equal $(1 - F(x))/f(x)$. Apparently, the sum of these two part is just equal to x, the value of the winning bidder, as what is shown in (22) and (23).

PROPOSITION 4 (efficient cutoffs):   *For a given mechanism $(\xi, \mathbf{M})$ with declining cutoff values, the expected social welfare, $W(\xi, \mathbf{M})$, is quasi-concave in ξ_t. The sequence of efficient cutoff values, $\{\xi_t^{**}\}_{1 \leq t \leq T}$, is the unique solution to*

$$(24) \qquad c_{M_{t+1}} = \int_{\xi_t^{**}}^1 (x - \xi_t^{**}) \, dF_{M_{t+1}}^{(1)}(x), \text{ for } 1 \leq t < T,$$

and $\xi_T^{**} = 0$ for $t = T$.

PROOF:

For $1 \leq t < T$, from (22), the derivative of $W(\xi, \mathbf{M})$ with respect to ξ_t is

$$\frac{\partial W}{\partial \xi_t} = f_{N_t}^{(1)}(\xi_t) \left[\int_{\xi_t}^1 (x - \xi_t) \, dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right] = f_{N_t}^{(1)}(\xi_t) \tilde{\eta}(\xi_t) = 0.$$

Note that $\tilde{\eta}(\xi_t)$ is decreasing in ξ_t, then W is quasi-concave in ξ_t, and the first order condition is also sufficient. Second, when $t = T$, the partial derivative of W with respect to ξ_t is $\frac{\partial W}{\partial \xi_T} = -\xi_T f_{N_t}^{(1)}(\xi_T) \leq 0$, and therefore $\xi_T^{**} = 0$. ∎

Not surprisingly, the efficient cutoff ξ_t^{**} in (24) also demonstrates a one-step-ahead property, as it depends only on the set M_{t+1} of bidders who will be invited in the next period. If we replace the true value v by its virtual value of $\psi(v)$, the formula for efficient cutoff value, (24), is identical to that for optimal cutoff value, (11). The connection between these two is clear: the virtual value $\psi(v)$ is the maximum revenue a seller can obtain from a truthful bidder, while the bidder's value v measures the social welfare if that bidder wins the product.

Similarly, we can define a function of efficient cutoff for searching a sample M of bidders, denoted by $\xi^{**}(M)$. For any $M \subseteq N$ with $M \neq \emptyset$, $\xi^{**}(M)$ is the unique solution to the equation of $c_M = \int_\xi^1 (x - \xi) \, dF_M^{(1)}(x)$, or equivalently,

$$(25) \qquad c_M = \int_{\xi^{**}(M)}^1 \left[1 - F_M^{(1)}(x) \right] dx.$$

From the definition, $\xi^{**}(M)$ measures the value of contacting the sample M of bidders, for the purpose of social welfare maximization. For example, if a seller has currently achieved a welfare level of ξ and it is equal to $\xi^{**}(M)$, then she will has no incentive to further contact the sample M of bidders at the cost of c_M, for the purpose of welfare maximization, as the net return is 0.

The following properties of $\xi^{**}(M)$ are straightforward: For two sets of bidders, $M, M' \subseteq N$, 1) if the cardinality $|M| = |M'|$ and $c_M < c_{M'}$, then $\xi^{**}(M) > \xi^{**}(M')$; 2) if $c_M = c_{M'}$ and $|M| < |M'|$, then $\xi^{**}(M) < \xi^{**}(M')$. The proof is similar to that of Corollary 6, and therefore is omitted here.

From (24) and (25), it is clear that the efficient cutoff value of $\xi_t^{**} = \xi^{**}(M_{t+1})$ for $1 \leq t < T$. Substituting (24) into (22), we then have the following expression of maximum

expected welfare, for a given family of bidder samples of $\mathbf{M}^T$.

LEMMA 7: *Given a family of bidder samples $\mathbf{M}^T$, the maximum expected social welfare is*

$$(26) \qquad W^{**}\left(\mathbf{M}^T\right) = W\left(\xi^{**}, \mathbf{M}\right) = \sum_{t=1}^{T} \int_{\xi^{**}(M_{t+1})}^{\xi^{**}(M_t)} \left[1 - F_{N_t}^{(1)}(x)\right] dx$$

B. Efficient Sampling Rule

To fix the idea of efficient sampling, as above, we consider the case of homogeneous bidders, by assuming that $c_i = c$ for all $i \in N$. In this homogeneous case, we denote $\xi^{**}(m)$ as the efficient cutoff value for searching a sample of m bidders, as given in (25). Intuitively, the sample size of m_t measures the seller's search intensity in period t.

LEMMA 8: *Suppose $c_i = c$ for all $i \in N$. The efficient cutoff value of $\xi^{**}(m)$ for inviting a sample M of bidders is strictly decreasing in its sample size of m. That is, for any two bidder samples, $M, M' \subseteq N$, if $m < m'$, then*

$$\xi^{**}(m) > \xi^{**}(m').$$

The following result is a direct implication of the above Lemma and the fact that, in an efficient search mechanism, the efficient cutoffs are declining over time. It shows the same result that, in an efficient search mechanism, it is optimal for the seller to sample increasingly more bidders when the deadline approaches.

PROPOSITION 5 (efficient sampling): *Suppose $c_i = c$ for all $i \in N$. If an sampling rule $\mathbf{M}^{**}$ is efficient, then the efficient sampling size is increasing in t, that is,*

$$m_t^{**} \leq m_{t+1}^{**}, \text{ for } t = 1, \cdots, T-1.$$

It is useful to make some comparison between the efficient and optimal cutoffs, as well as sample sizes. We have the following results.

COROLLARY 3: *Suppose $c_i = c$ for all $i \in N$. For a given sampling rule of $\mathbf{M}$, the optimal cutoff value is higher than the efficient one in each period, that is, for $1 \leq t \leq T$,*

$$\xi^{*}(m_t) > \xi^{**}(m_t).$$

PROOF:

1) For $1 \leq t < T$, ξ^* and ξ^{**} are given by (11) and (24) respectively. If we define

$$\tilde{\eta}(v) = \int_{v}^{1} (x-v) \, dF_{M_t}^{(1)}(x) - c_{M_t} \text{ and } \eta(v) = \int_{v}^{1} [\psi(x) - \psi(v)] \, dF_{M_t}^{(1)}(x) - c_{M_t},$$

then both $\tilde{\eta}(v)$ and $\eta(v)$ are decreasing in v. Note that

$$\eta(v) - \tilde{\eta}(v) = \int_v^1 \left[\frac{1 - F(v)}{f(v)} - \frac{1 - F(x)}{f(x)} \right] dF_{M_t}^{(1)}(x) > 0,$$

due to the IFR assumption. Finally, for $t = T$, we already know $r^* = \xi_T^* > \xi_T^{**} = 0$.

This result is reminiscent of the standard results in static auctions. In a symmetric static auction, the optimal reserve price r^* is set at the level such that its virtual value of $\psi(r^*) = 0$, which is strictly positive, while the efficient reserve price is simply 0. The same comparative result still holds in the case of sequential search auctions. When the sequence of bidder samples is given, the optimal cutoff value is strictly greater than the efficient one in each period.

Second, as the optimal and efficient cutoff functions of $\xi^*(m)$ and $\xi^{**}(m)$ are both strictly decreasing, we could define their inverse functions, denoted by $m^*(\xi_t)$ and $m^{**}(\xi_t)$ respectively, which roughly measure when the cutoff value ξ_t is given, what would be the corresponding optimal and efficient sample sizes.

PROPOSITION 6: *Suppose $c_i = c$ for all $i \in N$. For a given sequence of declining cutoff values ξ, the efficient sample size in each period is greater than the optimal one, that is,*

$$m^*(\xi_t) > m^{**}(\xi_t), \text{ for } t = 1, \cdots, T.$$

PROOF:

From (13) and (25), it follows that, for given ξ,

$$c = \int_\xi^1 \frac{1 - F^{m^*}(x)}{m^*} \cdot \psi'(x)\,dx = \int_\xi^1 \frac{1 - F^{m^{**}}(x)}{m^{**}}\,dx.$$

As $\psi'(x) > 1$ from the IFR assumption and $[1 - F^m(x)]/m$ is decreasing in m, we then get the result. ∎

The result shows that, when the sequence of declining cutoff values is given, there will be more bidders invited in each period in an optimal search auction than in an efficient one. As a result, the expected total number of participating bidders is also greater in an optimal search mechanism. This over-invitation result is also reported in the case of static search auctions, where a seller invites bidders at positive search costs and then runs a one-shot auction (Szech, 2011; Li, 2017; Li and Xu, 2016).

Let us consider the efficient search mechanism in a simple sample of 2-period with 3 long-lived bidders.

EXAMPLE 3: *Bidders are ex-ante homogeneous, with uniform value distribution on $[0, 1]$ and unit search cost of $c \in (0, 1/2)$. We denote the sampling rule by $\mathbf{m} = (m_1, m_2)$, where m_t is the bidder sample size in period t. We know that, under an efficient search procedure, it must be true that $m_1 \leq m_2$, and the candidates for efficient sampling rule is thus $\mathbf{m} = (1, 1)$ or $(1, 2)$, depending on the value of c. From formula (25), the efficient cutoff value for different sample size is given by:*
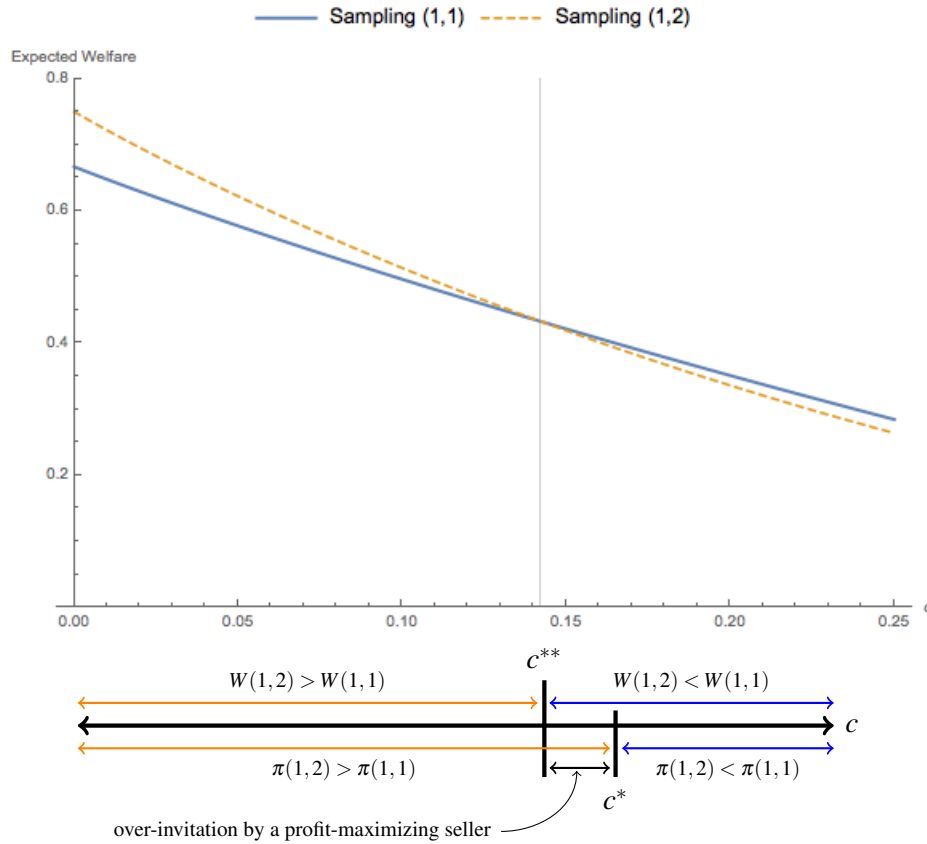
FIGURE 3. EXPECTED WELFARE IN THE 2-PERIOD 3-BIDDER PROBLEM

Note: See Example 3. When the invitation cost is lower than $c^{**} \approx 0.142$, the expected welfare from sequentially inviting one bidder and two bidders is greater than that from inviting one and one. If $c^{**} < c < 1/2$, however, inviting just one in both periods is better than inviting the three bidders over the two periods. It is also worth noting that the threshold c^{**} is lower than that c^* in Example 2. That is, for any cost between $c^{**} \approx 0.142$ and $c^* \approx 0.164$, the welfare-maximizing seller invites only one in the second period, while the profit-maximizing seller invites two in that period. This example confirms the tendency of over-invitation by profit-maximizing sellers, which causes inefficiency.

- *If $m = 1$, then $\xi^{**}(1)$ is the solution to $c = \frac{1}{2}(1-\xi)^2$, e.g., $\xi^{**}(1) = 1 - \sqrt{2c}$;*

- *If $m = 2$, then $\xi^{**}(2)$ is the solution to $2c = \frac{1}{3}(1-\xi)^2(2+\xi)$.*

Case 1: *If $c = \frac{1}{16}$, then $\xi^{**}(1) \approx 0.646$ and $\xi^{**}(2) \approx 0.622$. For a sampling rule of $(1,1)$, the expected social welfare, from (26), is*

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\,dx + \int_0^{\xi^{**}(1)} [1 - F^2(x)]\,dx \approx 0.556.$$

The expected welfare for a sampling rule of $(1,2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\,dx + \int_0^{\xi^{**}(2)} [1 - F^3(x)]\,dx \approx 0.593.$$

Therefore, the sampling rule of $(1,2)$ is optimal.

Case 2: *If $c = \frac{5}{24}$, then $\xi^{**}(1) \approx 0.355$ and $\xi^{**}(2) = 0.256$. For a sampling rule of $(1,1)$, the expected profit, from (18), is*

$$W(1,1) = \int_{\xi^{**}(1)}^{\xi^{**}(1)} [1 - F(x)]\,d\psi(x) + \int_0^{\xi^{**}(1)} [1 - F^2(x)]\,d\psi(x) \approx 0.340.$$

The expected profit for a sampling rule of $(1,2)$ is

$$W(1,2) = \int_{\xi^{**}(2)}^{\xi^{**}(1)} [1 - F(x)]\,d\psi(x) + \int_0^{\xi^{**}(2)} [1 - F^3(x)]\,d\psi(x) \approx 0.323.$$

Therefore, the sampling rule of $(1,1)$ now is efficient.

V. Optimal Search with Short-Lived Bidders

In this section, we will investigate the other important problem of finite sequential search with shorted-lived bidders. In contrast to a long-lived bidder, who will stay in the transactions until the deadline, a short-lived bidder, when invited, just participates in the stage transaction of that period, and then goes away. The case of short-lived bidders is analogous to sequential search with *no* recall.

We show that, with short-lived bidders, the optimal search mechanism is still featured by declining cutoff values and increasing search intensities (sample sizes). Therefore, the monotonicity results for optimal search mechanisms are robust, regardless of whether bidders are long-lived or short-lived. We also show that the outcomes of an optimal search mechanism, in this case, can also be implemented by a sequence of second price auctions.

Our first observation is that, when bidders are short-lived, the incentive problem for bidders becomes much simpler, and the cutoff value is always equal to the reserve price,

that is, $\xi_t = r_t$ for $t = 1, 2, \cdots, T$. This is because an invited short-lived bidder has but a single chance to submit a bid, and therefore, he will bid if and only if his value is greater than the reserve price of that period.

Similar to Section 3.3, we can also formulate a seller's sequential search problem with short-lived bidders as a DP problem. With the same notations, $\mathbf{M}^T$ is a family of bidder samples, and $N_t^c = \mathbf{M}^T \backslash N_t$ is the set of bidder samples that the seller has yet sampled till the end of period t. Suppose at the end of period t the seller's best offer is $\psi(v)$. The seller then faces the decision between the following two options: she can accept the current best offer and realize a revenue of $\psi(v)$, or she can decline it and continue searching in the next period. Different from the case of long-lived bidders, where a current best offer of $\psi(v)$ is always reclaimable in the future, now with short-lived bidders, the seller's fall back offer is always $0 = \psi(r^*)$, if she declines the current best offer and continues to search in the next period.

When bidders are short-lived, we denote $\hat{J}_t(v)$ as the value of having an offer of $\psi(v)$ in hand at the end of period t, which the seller can either accept or reject. It is obvious that $\hat{J}_{T+1}(v) = 0$ and $\hat{J}_T(v) = \max\{\psi(v), 0\}$. For $t < T$, the Bellman equation for this sequential search problem with short-lived bidders is thus

$$(27) \qquad \hat{J}_t(v) = \max_{M_{t+1} \in N_t^c} \left\{ \psi(v), -c_{M_{t+1}} + \mathbb{E}\hat{J}_{t+1}\left[\max\left\{r^*, X_{M_{t+1}}^{(1)}\right\}\right] \right\}.$$

Remember, in this case of sequential search with no recall, when the seller decides to continue searching, she already rejects the current best offer of $\psi(v)$ and her fall-back offer is thus $\psi(r^*) = 0$ in the next period, which corresponds to the virtual value of a bidder with value r^*.

As an analogue of (4), the Bellman equation (27) yields the recurrence relation between $\hat{\xi}_t^*$ and $\hat{\xi}_{t+1}^*$

$$(28) \qquad \psi(\hat{\xi}_t^*) = \max_{M_{t+1} \in N_t^c} \left\{ \int_0^1 \max\left\{ \psi(\hat{\xi}_{t+1}^*), \psi(x) \right\} dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right\},$$

where the left-hand side is the seller's value from stopping the search with the current cutoff $\hat{\xi}_t^*$ and the right-hand side refers her continuation value from keeping searching with the sample of M_{t+1}. It is important to note that the current cutoff $\hat{\xi}_t^*$ not only depends on the next period sample M_{t+1}, but also on the next period cutoff $\hat{\xi}_{t+1}^*$. In other words, the one-step-ahead property does not hold any more with short-lived bidders, as the current cutoff is determined recursively from the final period.

Rearranging the terms of (28), it follows that

$$\psi(\hat{\xi}_t^*) = \max_{M_{t+1} \in N_t^c} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \psi(x) dF_{M_{t+1}}^{(1)}(x) + \psi(\hat{\xi}_{t+1}^*) F_{M_{t+1}}^{(1)}(\hat{\xi}_{t+1}^*) - c_{M_{t+1}} \right\}$$

$$(29) \qquad = \max_{M_{t+1} \in N_t^c} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \left[\psi(x) - \psi(\hat{\xi}_{t+1}^*) \right] dF_{M_{t+1}}^{(1)}(x) - c_{M_{t+1}} \right\} + \psi(\hat{\xi}_{t+1}^*).$$

Therefore, compared to (11), when the next period sample M_{t+1} is given, the cutoff $\hat{\xi}_t^*$ with short-lived bidders is lower than ξ_t^* with long-lived bidders. The following proposition characterizes the optimal cutoffs with short-lived bidders.

PROPOSITION 7 (optimal cutoffs with short-lived bidders): *Given sample rule* **M**, *the sequence of optimal cutoffs $\hat{\xi}$ is recursively determined: $\psi(\hat{\xi}_T^*) = 0$ and for any $t < T$,*

$$\psi(\hat{\xi}_t^*) - \psi(\hat{\xi}_{t+1}^*) = \max_{M_{t+1} \in N_t^c} \left\{ \int_{\hat{\xi}_{t+1}^*}^1 \left[1 - F_{M_{t+1}}^{(1)}(x) \right] d\psi(x) - c_{M_{t+1}} \right\}.$$

We next show that the optimal search mechanism in this case is still featured by declining cutoff values and increasing search intensities. The proof of the following proposition uses the Principle of Optimality, which enables us to derive an optimal search rule by backward induction. At the end of period t, if a seller decides to continue searching, she needs to select the optimal bidder sample to contact in the next period, given that an optimal search rule will be followed in the following periods of $t+1, \cdots, T$. As before, to fix the idea of search intensity, we consider the case of homogeneous bidders, where $c_i = c$ for all $i \in N$.

PROPOSITION 8: *In an optimal search mechanism with short-lived bidders, the optimal cutoff value $\hat{\xi}_t^*$ is decreasing, and the optimal sample size $\hat{m}_t^*$ is increasing over time. That is, for all $t = 0, 1, \cdots, T-1$, we have*

$$\hat{\xi}_t^* > \hat{\xi}_{t+1}^*, \qquad \hat{m}_t^* \leq \hat{m}_{t+1}^*.$$

PROOF:

Denote $Z_{m_{t+1}}^{(1)} = \max\left\{ r^*, X_{m_{t+1}}^{(1)} \right\}$, and define

$$B_t = \max_{m_{t+1}} \left\{ E\hat{J}_{t+1} \left[Z_{m_{t+1}}^{(1)} \right] - m_{t+1} c \right\},$$

which is the continuation value of following an optimal search strategy after the end of period t. Temporarily, we define

(30) $\qquad \hat{J}_t(v) = \max_{m_{t+1}} \left\{ \psi(v), E\hat{J}_{t+1} \left[Z_{m_{t+1}}^{(1)} \right] - m_{t+1} c \right\} = \max\left\{ \psi(v), B_t \right\},$

which will later be shown is equivalent to (27). It is clear that $B_T = 0$, and for $t < T$,

$$B_t = \max_{m_{t+1}} \left\{ E\hat{J}_{t+1} \left[Z_{m_{t+1}}^{(1)} \right] - m_{t+1} c \right\} = \max_{m_{t+1}} \left\{ E\max\left\{ \psi\left(Z_{m_{t+1}}^{(1)} \right), B_{t+1} \right\} - m_{t+1} c \right\}.$$

When $t = T - 1$,

$$B_{T-1} = \max_{m_T} \left\{ E\hat{J}_T \left[Z_{m_T}^{(1)} \right] - m_T c \right\} = \max_{m_T} \left\{ E\max\left\{ \psi\left(Z_{m_T}^{(1)} \right), B_T \right\} - m_T c \right\} > 0 = B_T.$$

When $t = T - 2$, similarly,

$$
\begin{aligned}
B_{T-2} &= \max_{m_{T-1}} \left\{ \mathrm{E}\max \left\{ \psi\left(Z^{(1)}_{m_{T-1}}\right), B_{T-1} \right\} - m_{T-1}c \right\} \\
&> \max_{m_{T-1}} \left\{ \mathrm{E}\max \left\{ \psi\left(Z^{(1)}_{m_{T-1}}\right), B_T \right\} - m_{T-1}c \right\} = B_{T-1}.
\end{aligned}
$$

Continuing in this manner, we see that

$$
B_t > B_{t+1} \text{ for } t = 0, 1, \cdots, T-1.
$$

The result is then implied by the fact that the optimal cutoff satisfies $\psi\left(\hat{\xi}^*_t\right) = B_t$. Second, note that the optimal sample size m_t is the maximizer of

$$
\varsigma\left(m_t, \hat{\xi}^*_t\right) = \mathrm{E}\max \left\{ \psi\left(Z^{(1)}_{m_t}\right), B_t \right\} - m_t c.
$$

Simple transformation gives

$$
\varsigma\left(m_t, \hat{\xi}^*_t\right) = \mathrm{E}\max \left\{ \psi\left(X^{(1)}_{m_t}\right), \psi\left(\hat{\xi}^*_t\right) \right\} - m_t c = \psi\left(\hat{\xi}^*_t\right) + \int_{\hat{\xi}^*_t}^1 \left[\psi(x) - \psi\left(\hat{\xi}^*_t\right)\right] dF^{(1)}_{m_t}(x) - m_t c
$$

which is concave in m_t given the fact that the virtual value function, $\psi(x)$, is increasing and $\hat{\xi}^*_t$ is independent of m_t (Szech, 2011, Lemma 1). Furthermore,

$$
\varsigma\left(m_t + 1, \hat{\xi}^*_t\right) - \varsigma\left(m_t, \hat{\xi}^*_t\right) = \int_{\hat{\xi}^*_t}^1 F^{(1)}_{m_t}(x) \left[1 - F(x)\right] d\psi(x) - c
$$

is decreasing in $\hat{\xi}^*_t$. The optimization condition for m_t is

$$
\varsigma\left(m_t - 1, \hat{\xi}^*_t\right) - \varsigma\left(m_t, \hat{\xi}^*_t\right) \geq 0 > \varsigma\left(m_t + 1, \hat{\xi}^*_t\right) - \varsigma\left(m_t, \hat{\xi}^*_t\right).
$$

Given that $\hat{\xi}^*_t > \hat{\xi}^*_{t+1}$, then the concavity of $\varsigma\left(m_t, \hat{\xi}^*_t\right)$ in m_t then implies $\hat{m}^*_t \leq \hat{m}^*_{t+1}$. ∎

The optimal cutoff values are declining over time. The intuition is that, if an offer is good enough to be acceptable in period t, it should also be acceptable in period $t+1$ when there is one less chance for improvement. Alternatively, we can interpret the virtual value of $\psi\left(\hat{\xi}^*_t\right)$ as the outside option (reservation value) for a seller, which is determined by the continuation value of following an optimal search procedure. For any given sampling rule, the continuation value is obviously decreasing over time, as there is fewer trial opportunities for the seller to improve her payoff.

Now let us consider the implementation of the optimal search outcomes by a sequential auction. In our proof of the above Proposition, we solve for the optimal search mechanism by backward induction. For example, in the last period of T, it is obvious that the optimal reserve price $\hat{\xi}^*_T = r^*$, and there exists an optimal sample size of $\hat{m}^*_T$ that

maximizes the expected auction profit.[10] With the optimal solution of $\left(\hat{\xi}_T^*, \hat{m}_T^*\right)$ in the last period, we then have the continuation value of B_{T-1}, which is also the seller's reservation revenue at the end of period $T-1$. The optimal cutoff value in period $T-1$ such that $\psi\left(\hat{\xi}_{T-1}^*\right) = B_{T-1}$, with which we can derive the optimal sample size of $\hat{m}_{T-1}^*$, that the seller will sample in period $T-1$. Continuing this process, we then have the solution to the optimal mechanism of $\left(\hat{\boldsymbol{\xi}}^*, \hat{\mathbf{M}}^*\right)$. Simply setting a sequence of reserve prices $\hat{\mathbf{r}}^* = \hat{\boldsymbol{\xi}}^*$, then the sequential auction of $\left(\hat{\mathbf{r}}^*, \hat{\mathbf{M}}^*\right)$ implements the outcomes of an optimal sequential search mechanism with short-lived bidders.

When bidders are short-lived, we do not have an analytical solution of the optimal cutoff value of $\hat{\xi}_t^*$, as in the case of long-lived bidders. However, by comparing the two Bellman equations of (3), for long-lived bidders, and (27), for short-lived bidders, we can show that, for a given sampling rule $\mathbf{M}$, the optimal cutoff value for short-lived bidders is always lower than that for long-lived bidders. The intuition is that, when bidders are short-lived, a seller can not reclaim any offer declined in the previous periods, and therefore she is willing to accept lower cutoff prices, as her fall-back revenue is always lower than that with long-lived bidders.

PROPOSITION 9: *For a given sampling rule* $\mathbf{M}$*, the optimal cutoff value for short-lived bidders is smaller than that for long-lived bidders. That is, for* $0 \le t < T-1$,

$$\hat{\xi}_t^* < \xi_t^*,$$

and $\hat{\xi}_T^* = \xi_T^* = r^*$ *for* $t = T$.

PROOF:

For $t = T$, $\hat{J}_T(v) = \max\{\psi(v), 0\} = J_T(v)$, and therefore, $\hat{\xi}_T^* = \xi_T^* = r^*$. For $t = T-1$, for long-lived bidders, from (3),

$$J_{T-1}(v) = \max\left\{\psi(v), -c_{M_T} + \mathrm{E}J_T\left[\max\left\{v, X_{M_T}^{(1)}\right\}\right]\right\},$$

and $v \ge r^*$ as $\psi(v) \ge 0$. For short-lived bidders, from (27),

$$\hat{J}_{T-1}(v) = \max\left\{\psi(v), -c_{M_T} + \mathrm{E}\hat{J}_T\left[\max\left\{r^*, X_{M_T}^{(1)}\right\}\right]\right\}.$$

As $\hat{J}_T(v) = J_T(v)$ and both are increasing function, it is then clear that $J_{T-1}(v) \ge \hat{J}_{T-1}(v)$ with equality only when $v = 0$. Repeating this process, we then reaches the conclusion that $J_t(v) \ge \hat{J}_t(v)$, for $0 \le t < T-1$. The indifference condition for cutoff value then implies $\hat{\xi}_t^* < \xi_t^*$, for $0 \le t < T-1$. ∎

PROPOSITION 10: *Given a sequence of cutoff values* ξ *such that* $\xi_t > \xi_{t+1}$*, the optimal sample size for long-lived bidders is smaller than that for short-lived bidders in each period.*

---

[10]This optimization problem is a well-defined convex problem, given the IFR assumption (Szech, 2011; Li, 2017).
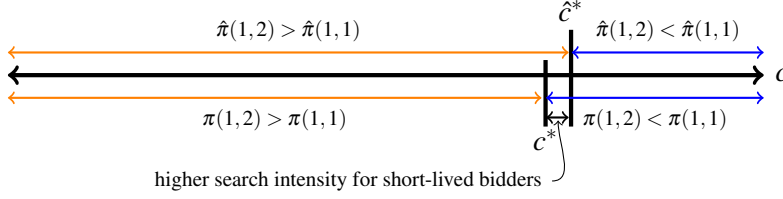
FIGURE 4. EXPECTED PROFIT WITH SHORT-LIVED BIDDERS

Note: See Example 4. When the invitation cost is lower than $\hat{c}^* \approx 0.167$, it is optimal to invite one bidder in the first period and two the remaining bidders in the second period. If $c > \hat{c}^*$, however, it is optimal to invite just one bidder sequentially in both periods. Compared to Example 2, if the cost is in between $c^* \approx 0.164$ and $\hat{c}^* \approx 0.167$, the seller invites more bidders in the second period when they are short-lived.

PROOF:

Recall the condition (4) for long-lived bidders and the recurrence equations (28) for short-lived bidders. Given a sequence of cutoff values ξ such that $\xi_t > \xi_{t+1}$, the above equations define the inverse real-value functions of $m_{t+1}^*(\xi_t)$ for long-lived bidders and $\hat{m}_{t+1}^*(\xi_t, \xi_{t+1})$ for short-lived bidders. That is, for given ξ, $m^*(\xi_t)$ and $\hat{m}^*(\xi_t, \xi_{t+1})$ are respectively the optimal sample sizes for long-lived and short-lived bidders in period $t+1$. Our objective is to show $m_{t+1}^*(\xi_t) < \hat{m}_{t+1}^*(\xi_t, \xi_{t+1})$. We can define a new function

$$\eta(m, \xi) = \int_0^1 \max\{\psi(\xi), \psi(x)\} \, dF^m(x) - mc$$

which is strictly concave in m (Szech, 2011), and obeys single-crossing difference in (m, ξ) given that, for $m' > m$, $\eta(m'', \xi) - \eta(m', \xi)$ is decreasing in ξ. The well-known result of Milgrom and Shannon (1994)(Theorem 4) gives that

$$\tilde{m}(\xi) \equiv \arg\max_{\xi} \eta(m, \xi)$$

is strictly decreasing in ξ, and hence $\tilde{m}(\xi_t) < \tilde{m}(\xi_{t+1})$. In addition, from (4), it follows

$$\begin{aligned} \psi(\xi_t) &= \int_0^1 \max\{\psi(\xi_t), \psi(x)\} \, dF^{m^*(\xi_t)}(x) - m^*(\xi_t) c \\ &\leq \int_0^1 \max\{\psi(\xi_t), \psi(x)\} \, dF^{\tilde{m}(\xi_t)}(x) - \tilde{m}(\xi_t) c \end{aligned}$$

Therefore,

$$m^*(\xi_t) \leq \tilde{m}(\xi_t) < \tilde{m}(\xi_{t+1}) = \hat{m}^*(\xi_t, \xi_{t+1}),$$

where the last equality is implied by (28). ∎

We now consider a simple 2-period example to illustrate how to derives the optimal search procedure in the case of short-lived bidders.

EXAMPLE 4: *Bidders are ex-ante homogeneous, with uniform value distribution on*

[0,1] and unit search cost of $c > \underline{c} \approx 0.047$.[11] We denote the sampling rule by $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2)$, where $\hat{m}_t$ is the bidder sample size in period t. We consider two candidates for optimal sampling rule, $\hat{\mathbf{m}} = (1,1)$ and $(1,2)$.

We define B_t, $t = 1, 2$, as the continuation value of following an optimal search procedure after the end of period t, and the optimal cutoff value $\hat{\xi}_t^*$ satisfies $\psi\left(\hat{\xi}_t^*\right) = B_t$, as implied by the Bellman equation of (27). It is clear that $B_2 = 0$ and $\hat{\xi}_2^* = r^*$; and

$$(31) \qquad B_1 = \max_{\hat{m}_2} \left\{ \int_{r^*}^1 \psi(x) \, dF^{\hat{m}_2}(x) - \hat{m}_2 c \right\},$$

where the virtual value function $\psi(x) = 2x - 1$. Following an optimal search procedure of $\left\{ \left(\hat{\xi}_1^*, r^*\right), (\hat{m}_1^*, \hat{m}_2^*) \right\}$, the expected profit is

$$\hat{\pi}^* = \left[\int_{\hat{\xi}_1^*}^1 \psi(x) \, dF^{\hat{m}_1^*}(x) - \hat{m}_1^* c \right] + F^{\hat{m}_1^*}\left(\hat{\xi}_1^*\right) \left[\int_{r^*}^1 \psi(x) \, dF^{\hat{m}_2^*}(x) - \hat{m}_2^* c \right].$$

Case 1: $\underline{c} < c \le \hat{c}^* = 1/6$
   $\hat{m}_2^* = 2$ *maximizes (31) with* $B_1 = \frac{5}{12} - 2c$ *and* $\hat{\xi}_1^* = \frac{17}{24} - c$

Case 2: $\hat{c}^* < c < 1/4$
   $\hat{m}_2^* = 1$ *maximizes (31) with* $B_1 = \frac{1}{4} - c$ *and* $\hat{\xi}_1^* = \frac{5}{8} - \frac{1}{2}c$

This example also verifies the result of Proposition 8, that is, for given sampling rule, the optimal cutoff value with short-lived bidders is smaller that with long-lived bidders, in each period. For instance, for $c = \frac{1}{16}$, it follows $\hat{m}_2^* = 2$, $B_1 = \frac{7}{24}$ and

$$\hat{\xi}_1^* = \frac{31}{48} \approx 0.646 < \xi_1^* \approx 0.738,$$

and for $c = \frac{5}{24}$, it follows $\hat{m}_2^* = 1$, $B_1 = \frac{1}{24}$, and

$$\hat{\xi}_1^* = \frac{25}{48} \approx 0.521 < \xi_1^* \approx 0.544,$$

where ξ_1^*'s are derived in Example 2.

VI.   Conclusion

This paper studies sequential search auctions with a deadline, where a seller needs to allocate a non-divisible product among a number of bidders by a finite deadline, and to contact a bidder, she needs to incur a positive cost. We show that an optimal search auction is featured by declining reserve prices and increasing search intensities (sample

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[11]For a very low $c < \underline{c}$, it is optimal to invite all the three bidders in the first period, as $\hat{m}_2^* = 3$ maximizes (31).

sizes). Our results help explain why, in practice, many selling processes seem not competitive, where no obvious competition among bidders is observed, such as in M&As.

To confirm the robustness of the monotonicity results, in addition to an optimal search auction, this paper also considers an efficient mechanism as well as the cases of both long-lived and short-lived bidders. We find that both both the optimal reserve prices and the optimal search intensities are higher than the efficient ones. This identifies a source on inefficiency in sequential auctions as the profit-maximizing seller has an incentive to over-invite bidders, compared to the efficient level. In the case with short-lived bidders, we show that the optimal reserve prices are lower but the optimal search intensities are higher than those with long-lived bidders. This implies that the seller tends to set a lower reserve price and to spend a higher search intensity in each period when she faces short-lived bidders.

This paper develops a framework for studying finite sequential search problems in strategic environments, e.g., in auctions as studied in this paper. We believe this framework can be applied to a large variety of related problems, such as sequential matching in marriage markets, sequential contests, or job recruiting by a deadline. These extensions may be left for future research.

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MATHEMATICAL APPENDIX

PROOF OF LEMMA 1:

It is obvious that $\bar{U}_t(0) = \bar{U}_{t+1}(0) = 0$. Since the product must be sold at a positive probability, there exists v° such that $\bar{U}_t(v^\circ) > \bar{U}_{t+1}(v^\circ)$. First, we show $\bar{U}_t(v) > \bar{U}_{t+1}(v)$ for any $v \geq v^\circ$. Suppose, for a contradiction, that there exists $\bar{U}_t(v) = \bar{U}_{t+1}(v)$ for some $v \geq v^\circ$. Let $\bar{v} = \min\{v \geq v^\circ \mid \bar{U}_t(v) = \bar{U}_{t+1}(v)\}$, which is well-defined as $\bar{U}_t$ and $\bar{U}_{t+1}$ are continuous. Then for any $\tilde{v} \in [v^\circ, \bar{v})$, it must be $\bar{U}_t(\tilde{v}) > \bar{U}_{t+1}(\tilde{v})$ and hence $\bar{U}'_t(\tilde{v}) = F^{(1)}_{N_t \setminus \{i\}}(\tilde{v})$, which is in turn strictly greater than $F^{(1)}_{N_{t+1} \setminus \{i\}}(\tilde{v}) \geq \bar{U}'_{t+1}(\tilde{v})$. It contradicts to the continuity of $\bar{U}_t$ and $\bar{U}_{t+1}$ and hence $\bar{U}_t(v) > \bar{U}_{t+1}(v)$ for any $v \geq v^\circ$. Then, $\xi_t = \max\{v \mid \bar{U}_t(v) = \bar{U}_{t+1}(v)\}$ is uniquely defined and the standard payoff equivalence argument yields the bidder's payoff function as (5). ∎

PROOF OF LEMMA 2:

For $t < T$, a bidder with the cutoff value ξ_t is indifferent between bidding and waiting, and therefore $\bar{U}_t(\xi_t) = U^b_t(\xi_t) = \bar{U}_{t+1}(\xi_t)$. As $\xi_t \geq \xi_{t+1}$, he then prefers bidding to waiting in period $t+1$, and hence $\bar{U}_{t+1}(\xi_t) = U^b_{t+1}(\xi_t)$. It then follows that, for $t < T$,

$$
\begin{aligned}
U^b_{t+1}(\xi_t) &= F^{(1)}_{N_{t+1}\setminus\{i\}}(\xi_{t+1})\left(\xi_t - r_{t+1}\right) + \mathbb{I}_{\{\xi_t \geq \xi_{t+1}\}} \int_{\xi_{t+1}}^{\xi_t} (\xi_t - x)\, dF^{(1)}_{N_{t+1}\setminus\{i\}}(x) \\
&= F^{(1)}_{N_{t+1}\setminus\{i\}}(\xi_{t+1})\left(\xi_{t+1} - r_{t+1}\right) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1}\setminus\{i\}}(x)\, dx,
\end{aligned}
$$

and $U^b_t(\xi_t) = F^{(1)}_{N_t\setminus\{i\}}(\xi_t)\left(\xi_t - r_t\right)$. We then prove the result of (6). Second, $\xi_T = r_T$ obviously, as $\bar{U}_{T+1}(v) = 0$.

We next show the reserve prices $\{r_t\}_{1 \le t \le T}$ are also decreasing in t. From (6),

$$
\begin{aligned}
F^{(1)}_{N_t \setminus \{i\}}(\xi_t)(\xi_t - r_t) &= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_{t+1})(\xi_{t+1} - r_{t+1}) + \int_{\xi_{t+1}}^{\xi_t} F^{(1)}_{N_{t+1} \setminus \{i\}}(x)dx \\
&\le F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_{t+1} - r_{t+1}) + F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_t - \xi_{t+1}) \\
&= F^{(1)}_{N_{t+1} \setminus \{i\}}(\xi_t)(\xi_t - r_{t+1}).
\end{aligned}
$$

The result then implies $r_t \ge r_{t+1}$, as desired. ∎

PROOF OF LEMMA 3:

In the stage auction of period t, there are total $N_t = N_{t-1} \cup M_t$ bidders. Among them, M_t bidders are strong whose values are independent draws from F on $[0,1]$, and the other N_{t-1} bidders are weak, whose values are independent draws from the truncated distribution of $F(v|\xi_{t-1}) = \Pr(V \le v | V \le \xi_{t-1})$. The reserve price is r_t, and bidders' cutoff value for submitting bid is ξ_t, with $\xi_t > r_t$ for $t < T$, as shown in Proposition 2.

We need to introduce some new notations here. We denote the truncated distribution by $F(x|\xi_{t-1}) = F(x)/F(\xi_{t-1})$. Moreover, let $G^{(k)}_{N_t}$, $k = 1,2$, denote the distribution function of the k-th highest order statistics of the values of the N_t bidders'. Based on the properties of order statistics, we have the following expressions of $G^{(k)}_{N_t}$.

$$
\begin{aligned}
G^{(1)}_{N_t}(x) &= F^{(1)}_{N_{t-1}}(v|\xi_{t-1})F^{(1)}_{M_t}(v), \\
(A1)\quad G^{(2)}_{N_t}(x) &= F^{(1)}_{N_{t-1}}(v|\xi_{t-1})F^{(2)}_{M_t}(v) + n_{t-1}\bar{F}(x|\xi_{t-1})F^{(1)}_{N_{t-1}-1}(v|\xi_{t-1})F^{(1)}_{M_t}(v),
\end{aligned}
$$

where $\bar{F}(x|\xi_{t-1}) = 1 - F(x|\xi_{t-1})$ is the survival function. The expected revenue of the stage auction in period t is thus

$$
R_t(N_t) = r_t\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right] + \int_{\xi_t}^{1} x\, dG^{(2)}_{N_t}(x)
$$

It is helpful to do the following transformation,

(A2)
$$
R_t(N_t) = \left\{\xi_t\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right] + \int_{\xi_t}^{1} x\, dG^{(2)}_{N_t}(x)\right\} - (\xi_t - r_t)\left[G^{(2)}_{N_t}(\xi_t) - G^{(1)}_{N_t}(\xi_t)\right],
$$

where the part in the curly braces is the expected revenue of a one-shot auction with a reserve price of ξ_t. From Myerson (1981) and Kirkegaard (2012), it is equal to

(A3)
$$
\begin{aligned}
&F^{(1)}_{N_{t-1}}(\xi_t|\xi_{t-1})\int_{\xi_t}^{1} \psi(x)\, dF^{(1)}_{M_t}(x) \\
&\quad + \int_{\xi_t}^{\xi_{t-1}}\left[\psi(v|\xi_{t-1})F^{(1)}_{M_t}(v) + \int_{v}^{1} \psi(x)\, dF^{(1)}_{M_t}(x)\right] dF^{(1)}_{N_{t-1}}(v|\xi_{t-1})
\end{aligned}
$$

where $\psi(v|\xi_{t-1})$ is the virtual value function of the N_{t-1} weak bidders. Substituting

$\psi(v|\xi_{t-1}) = \psi(v) + \frac{\bar{F}(\xi_{t-1})}{f(v)}$ into (A3) and integrating by parts, we then have the expected revenue of (A3) is equal to

$$\text{(A4)} \quad \int_{\xi_t}^{\xi_{t-1}} \psi(x)\, d\frac{F_{N_t}^{(1)}(x)}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} + \int_{\xi_{t-1}}^{1} \psi(x)\, dF_{M_t}^{(1)}(x) + \frac{n_{t-1}\bar{F}(\xi_{t-1})}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} \int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)\, dx$$

Second, from the cutoff condition of (6) for bidders' equilibrium strategies, we have

$$\text{(A5)} \quad \int_{\xi_t}^{\xi_{t-1}} F_{N_t-1}^{(1)}(x)\, dx = F_{N_{t-1}-1}^{(1)}(\xi_{t-1})\,(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)\,(\xi_t - r_t).$$

Moreover, the the property of order statistics implies that

$$\text{(A6)} \quad G_{N_t}^{(2)}(\xi_t) - G_{N_t}^{(1)}(\xi_t) = m_t\bar{F}(\xi_t)\frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}}^{(1)}(\xi_{t-1})} + n_{t-1}\bar{F}(x|\xi_{t-1})\frac{F_{N_t-1}^{(1)}(\xi_t)}{F_{N_{t-1}-1}^{(1)}(\xi_{t-1})}.$$

Substituting the results of (A3)-(A6) into (A2), we then have the *ex-ante* expected stage revenue in period t:

$$\begin{aligned}
F_{N_{t-1}}^{(1)}(\xi_{t-1})\, R_t(N_t) &= \int_{\xi_t}^{\xi_{t-1}} \psi(x)\, dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1})\int_{\xi_{t-1}}^{1} \psi(x)\, dF_{M_t}^{(1)}(x) \\
&\quad + n_{t-1}\bar{F}(\xi_{t-1})\left[F_{N_{t-1}-1}^{(1)}(\xi_{t-1})\,(\xi_{t-1} - r_{t-1}) - F_{N_t-1}^{(1)}(\xi_t)\,(\xi_t - r_t)\right] \\
&\quad - (\xi_t - r_t)\left[m_t\bar{F}(\xi_t)F_{N_t-1}^{(1)}(\xi_t) + n_{t-1}\,(F(\xi_{t-1}) - F(\xi_t))\,F_{N_t-1}^{(1)}(\xi_t)\right] \\
&= \int_{\xi_t}^{\xi_{t-1}} \psi(x)\, dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1})\int_{\xi_{t-1}}^{1} \psi(x)\, dF_{M_t}^{(1)}(x) \\
&\quad + n_{t-1}\bar{F}(\xi_{t-1})F_{N_{t-1}-1}^{(1)}(\xi_{t-1})\,(\xi_{t-1} - r_{t-1}) - n_t\bar{F}(\xi_t)F_{N_t-1}^{(1)}(\xi_t)\,(\xi_t - r_t).
\end{aligned}$$

Summing all of them together, we then get the result of (9). ∎

PROOF OF COROLLARY 1:
Result 1) is straightforward from (13), as $F_M^{(1)}(x) = F_{M'}^{(1)}(x) = F^{|M|}(x)$. For result 2), as $c_M = c_{M'}$, then from (13),

$$\int_{\xi^*(M)}^{1} \left[1 - F_M^{(1)}(x)\right] d\psi(x) = \int_{\xi^*(M')}^{1} \left[1 - F_{M'}^{(1)}(x)\right] d\psi(x) > \int_{\xi^*(M')}^{1} \left[1 - F_M^{(1)}(x)\right] d\psi(x)$$

where the inequality is due to $F_M^{(1)}(x) > F_{M'}^{(1)}(x)$. It then follows that $\xi^*(M') > \xi^*(M)$ as $\int_{\xi^*(M)}^{\xi^*(M')} \left[1 - F_M^{(1)}(x)\right] d\psi(x) > 0$. ∎

PROOF OF COROLLARY 2:

The definition of MPS implies $\int_0^v \frac{\partial F(x,\sigma)}{\partial \sigma} dx \geq 0$, and $\frac{\partial F(0,\sigma)}{\partial \sigma} = \frac{\partial F(1,\sigma)}{\partial \sigma} = 0$ for all σ. From (13), it follows from simple calculation that

$$c_M = \mathbb{E}\psi\left(V_M^{(1)}\right) - \psi(\xi^*(M;\sigma)) + \int_0^{\xi^*(M;\sigma)} F_M^{(1)}(x;\sigma) \, d\psi(x)$$

where $\mathbb{E}\psi\left(V_M^{(1)}\right)$ is constant across σ by definition of MPS. Taking partial derivative with respect to σ and by re-arrangement, we then arrives at

$$\left[1 - F_M^{(1)}(\xi^*;\sigma)\right] \psi'(\xi^*) \frac{\partial \xi^*}{\partial \sigma} = \int_0^{\xi^*(M;\sigma)} \frac{\partial F_M^{(1)}(x;\sigma)}{\partial \sigma} d\psi(x),$$

and therefore $\partial \xi^* / \partial \sigma > 0$ as $\partial F_M^{(1)} / \partial \sigma > 0$ due to MPS and $\psi' > 0$ due to IFR. ∎

PROOF OF LEMMA 4:
From (13) and (11), we have $\xi^*(M_{t+1}) = \xi_t^*$ for $1 \leq t < T$, and define $\xi^*(M_{T+1}) \equiv \xi_T^*$ Substituting (11) into (9), we then get

$$
\begin{aligned}
\pi^*(\mathbf{M}) &= \left[\int_{\xi_1^*}^{\xi_0} \psi(x) \, dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \sum_{t=2}^{T} \left[\int_{\xi_t^*}^{\xi_{t-1}^*} \psi(x) \, dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}^*) \int_{\xi_{t-1}^*}^{1} \psi(\xi_{t-1}^*) \, dF_{M_t}^{(1)}(x)\right] \\
&= \left[\int_{\xi_1^*}^{1} \psi(x) \, dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \sum_{t=2}^{T} \left[\psi(\xi_{t-1}^*) F_{N_{t-1}}^{(1)}(\xi_{t-1}^*) - \psi(\xi_t^*) F_{N_t}^{(1)}(\xi_t^*) - \int_{\xi_t^*}^{\xi_{t-1}^*} F_{N_t}^{(1)}(x) \, d\psi(x)\right] \\
&= \left[\int_{\xi_1^*}^{1} \psi(x) \, dF_{N_1}^{(1)}(x) - c_{M_1}\right] \\
&\quad + \psi(\xi_1^*) F_{N_1}^{(1)}(\xi_1^*) - \psi(\xi_1^*) + \sum_{t=2}^{T} \int_{\xi_t^*}^{\xi_{t-1}^*} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x) \\
&= \int_{\xi^*(M_2)}^{\xi^*(M_1)} \left[1 - F_{N_1}^{(1)}(x)\right] d\psi(x) + \sum_{t=2}^{T} \int_{\xi^*(M_{t+1})}^{\xi^*(M_t)} \left[1 - F_{N_t}^{(1)}(x)\right] d\psi(x).
\end{aligned}
$$

For the last equality, we apply the definition that $c_{M_1} = \int_{\xi^*(M_1)}^{1} \left[1 - F_{M_1}^{(1)}(x)\right] d\psi(x)$. ∎

PROOF OF LEMMA 6:
The conditional expected social welfare in period t is

$$W_t(N_t) = \int_{\xi_t}^{\xi_{t-1}} x \, dG_{N_t}^{(1)}(x) + \int_{\xi_{t-1}}^{1} x \, dF_{M_t}^{(1)}(x).$$

where $G_{N_t}^{(1)}(x) = F_{N_{t-1}}^{(1)}(x|\xi_{t-1})F_{M_t}^{(1)}(x)$. Summing up all the terms of $F_{N_{t-1}}^{(1)}(\xi_{t-1})W_t(N_t)$, we then get the result of (22). ∎

PROOF OF LEMMA 7:

From (25) and (24), we have $\xi^{**}(M_{t+1}) = \xi_t^{**}$ for $1 \leq t < T$. For $t = T$, we define $\xi^{**}(M_{T+1}) = \xi_T^{**} = 0$. We then have

$$
\begin{aligned}
W^{**} &= \left[\int_{\xi_1^{**}}^{\xi_0} x\,dF_{N_1}^{(1)}(x) - c_{M_1}\right] + \sum_{t=2}^{T}\left[\int_{\xi_t^{**}}^{\xi_{t-1}^{**}} x\,dF_{N_t}^{(1)}(x) + F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**})\int_{\xi_{t-1}^{**}}^{1}\xi_{t-1}^{**}\,dF_{M_t}^{(1)}(x)\right] \\
&= \left[\int_{\xi_1^{**}}^{1} x\,dF_{N_1}^{(1)}(x) - c_{M_1}\right] + \sum_{t=2}^{T}\left[\xi_{t-1}^{**}F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**}) - \xi_t^{**}F_{N_t}^{(1)}(\xi_t^{**}) - \int_{\xi_t^{**}}^{\xi_{t-1}^{**}} F_{N_t}^{(1)}(x)\,dx\right] \\
&= \left[\xi_1^{**} - \xi_1^{**}F_{N_1}^{(1)}(\xi_1^{**}) + \int_{\xi_1^{**}}^{\xi^{**}(M_1)}\left[1 - F_{N_1}^{(1)}(x)\right]dx\right] \\
&\quad + \sum_{t=2}^{T}\left[\xi_{t-1}^{**}F_{N_{t-1}}^{(1)}(\xi_{t-1}^{**}) - \xi_t^{**}F_{N_t}^{(1)}(\xi_t^{**}) + \int_{\xi_t^{**}}^{\xi_{t-1}^{**}}\left[1 - F_{N_t}^{(1)}(x)\right]dx - (\xi_{t-1}^{**} - \xi_t^{**})\right] \\
&= \sum_{t=1}^{T}\int_{\xi^{**}(M_{t+1})}^{\xi^{**}(M_t)}\left[1 - F_{N_t}^{(1)}(x)\right]dx,
\end{aligned}
$$

where in the third equality, we substitute $c_{M_1} = \int_{\xi^{**}(M_1)}^{1}\left[1 - F_{M_1}^{(1)}(x)\right]dx$. ∎

PROOF OF LEMMA 8:

We would like to show $\xi^{**}(m+1) < \xi^{**}(m)$. For $m+1$ bidders, we have

$$
(m+1)c = \frac{m+1}{m}\int_{\xi^{**}(m+1)}^{1}\left[x - \xi^{**}(m+1)\right]F(x)\,dF^m(x).
$$

Therefore,

$$
mc = \int_{\xi^{**}(m+1)}^{1}\left[x - \xi^{**}(m+1)\right]F(x)\,dF^m(x) = \int_{\xi^{**}(m)}^{1}\left[x - \xi^{**}(m)\right]dF^m(x).
$$

As $F(x) < 1$ for $x \in [0,1)$, the only way to get the equality hold is that $\xi^{**}(m+1) < \xi^{**}(m)$. ∎