Nondictatorial Arrovian Social Welfare Functions, Simple Majority Rule, and Integer Programming

Francesca Busetto, Giulio Codognaton and Simone Tonin

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Abstract

In this paper, we use the linear programming approach to mechanism design, first introduced by Sethuraman et al. (2003) and then systematized by Vohra (2011), to analyze nondictatorial Arrovian social welfare functions with and without ties. First, we provide a new and simpler proof of Theorem 2 in Kalai and Muller (1977), which characterizes the domains admitting nondictatorial Arrovian social welfare functions without ties. Then, we show that a domain containing an inseparable ordered pair admits nondictatorial Arrovian social welfare functions with ties, thereby strengthening a result previously obtained by Kalai and Ritz (1978). Finally, we propose a reformulation of the simple majority rule in the framework of integer programming with an odd or even number of agents. We use this reformulation to recast some celebrated theorems, proved by Arrow (1963), Sen (1966), and Inada (1969), which provide conditions guaranteeing that the simple majority rule is a nondictatorial Arrovian social welfare function.
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1   Introduction

Vohra (2011) based his monograph on mechanism design on linear programming. He claimed that this approach has basically three advantages: simplic-

ity, unity, and reach, meaning, respectively, that it may simplify arguments, unify disparate results, and solve problems which are beyond the reach of other approaches. The first chapter of the book starts with a "genuflection," in the words of the author, towards Arrow's impossibility theorem (see Arrow (1963)). In this chapter, the author basically refers to the integer programming approach to Arrovian Social Welfare Functions (ASWFs) which was introduced by Sethuraman et al. (2003). In particular, Sethuraman et al. (2003) developed Integer Programs (IPs) in which variables assume values only in the set $\{0, 1\}$. These IPs were inspired by the characterization of decomposable domains introduced by Kalai and Muller (1977) and they allowed Sethuraman et al. (2003) to establish a one-to-one correspondence, on domains of antisymmetric preference orderings, between the set of feasible solutions of a binary IP and the set of ASWFs without ties. Subsequently, Busetto et al. (2015) generalized the approach proposed by Sethuraman et al. (2003), specifying IPs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$, called ternary IPs, and they established a one-to-one correspondence between the set of feasible solutions of a ternary IP and the set of ASWFs with and without ties.

In this paper, we use the evoked advantages of the linear programming approach to analyze nondictatorial ASWFs. Our analysis is, to some extent, complementary to that undertaken by Sethuraman et al. (2006) about anonymous monotonic ASWFs in an integer programming framework. First, we reconsider the IPs proposed by Sethuraman et al. (2003), we show that one of them exhibits some redundant constraints, and we repropose the IPs introduced by Busetto et al. (2015) which amend the redundancies and allow for ternary solutions. We then use the generalized IPs proposed by Busetto et al. (2015) to obtain a new and simpler proof of Theorem 2 in Kalai and Muller (1977) for nondictatorial ASWFs without ties. To this end, we use the notion of decomposability introduced by Busetto et al. (2015) to eliminate the redundancy of some conditions of the definition of decomposability proposed by Kalai and Muller (1977) which parallels the redundancy of the constraints in the IPs formalized by those authors mentioned above. Moreover, we restate the notion of a strictly decomposable domain, introduced by Busetto et al. (2015), and their characterization theorem, establishing that a domain of antisymmetric preference orderings admits nondictatorial ASWFs with ties if and only if it is strictly decomposable. We then reconsider the notion of a domain containing an inseparable ordered pair introduced, in an unpublished paper, by Kalai and Ritz (1978). They showed that such a domain must be decomposable, and consequently, it always admits nondic-

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tatorial ASWFs without ties. Domains containing an inseparable ordered pair were studied, among others, by Kalai and Ritz (1980), Kim and Roush ((1980),(1981)), Blair and Muller (1983), Ritz ((1983), (1985)), Muller and Satterthwaite (1985). We strengthen the result obtained by Kalai and Ritz (1978), showing that a domain containing an inseparable ordered pair must be strictly decomposable and consequently it must also admit nondictatorial ASWFs with ties. Finally, we consider a reformulation of the Simple Majority Rule (SMR) in the framework of integer programming with an odd or even number of agents. There is a huge literature about the domains admitting the SMR as a nondictatorial ASWF (see Gaertner (2001) for a survey). We first restate the integer programming version, provided by Sethuraman et al. (2003), of a theorem proved by Sen (1966), which shows that, when the number of agents is odd, a necessary and sufficient condition for the SMR to be a nondictatorial ASWF is that it is defined on a domain which does not contain a Condorcet triple. We then provide a short proof, based on integer programming, of the celebrated possibility theorem, first proved by Arrow (1963), which shows that, when the number of agents is odd, the SMR is an nondictatorial ASWF if it is defined on the domain of single-peaked preference orderings. Both of these theorems characterize the SMR as a nondictatorial ASWF without ties. Therefore, we straightforwardly show that the domains which do not contain a Condorcet triple or which are single-peaked are both decomposable. Inada (1969) proposed some sufficient conditions for the SMR to be a nondictatorial ASWF when the number of agents is odd or even. In particular, we use the integer programming to restate and prove a result, provided by Inada (1969), which shows that, for any number of agents, the SMR is a nondictatorial ASWF if it is defined on an echoic domain. When the number of agents is even, a SMR defined on an echoic domain is a nondictatorial ASWF with ties and this implies that an echoic domain must be strictly decomposable.

The paper is organized as follows. In Section 2, we introduce the notation and the basic definitions. In Section 3, we introduce and discuss the IPs and their correspondence with the ASWFs. In Section 4, we provide a new and shorter proof of Theorem 2 in Kalai and Muller (1977) and a stronger version of the theorem proved by Kalai and Ritz (1978). In Section 5, we use integer programming to prove three classical results on domains on which the SMR is a nondictatorial ASWF and we compare those domains with decomposable and strictly decomposable domains. In Section 6, we draw some conclusions.

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2 Notation and definitions

Let E be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of E, denoted by n. Elements of E are called agents.

Let $\mathcal{E}$ be the collection of all subsets of E. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

Let $\mathcal{R}$ be the set of all the complete and transitive binary relations on $\mathcal{A}$, called preference orderings.

Let Σ be the set of all antisymmetric preference orderings.

Let Ω denote a subset of Σ such that $|\Omega| \geq 2$. An element of Ω is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if x is ranked above y under $\mathbf{p}$.

Given $\mathbf{p} \in \Sigma$, let $\mathbf{p}^{-1}$ denote an antisymmetric preference ordering such that, for each $(x, y) \in \mathcal{A}^2$, $x\mathbf{p}y$ if and only if $y\mathbf{p}^{-1}x$

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let TR denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let NTR denote the set of nontrivial pairs.

Let Ω^n denote the n-fold Cartesian product of Ω. An element of Ω^n is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on Ω is a function $f : \Omega^n \to \mathcal{R}$.

f is said to be "without ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) = \emptyset$.

f is said to be "with ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) \neq \emptyset$.

Given $\mathbf{P} \in \Omega^n$, let $P(f(\mathbf{P}))$ and $I(f(\mathbf{P}))$ be binary relations on $\mathcal{A}$. We write $xP(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ but not $yf(\mathbf{P})x$ and $xI(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ and $yf(\mathbf{P})x$.

A SWF on Ω, f, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_iy$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$.

A SWF on Ω, f, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P}' \in \Omega^n$, $x\mathbf{p}_iy$ if and only if $x\mathbf{p}'_iy$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P}')y$, and, $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P}')x$.

An Arrovian Social Welfare Function (ASWF) on Ω is a SWF on Ω, f, which satisfies PO and IIA.

An ASWF on Ω, f, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_j y$ implies $xP(f(\mathbf{P}))y$. f is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, \frac{1}{2}, 1\}$.

An Integer Program (IP) on Ω consists of a set of linear constraints, related to the preference orderings in Ω, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let d denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on Ω. d is said to be a binary solution if variables $d_S(x, y)$ reduce to assume values in the set $\{0, 1\}$, for all $(x, y) \in NTR$, and for all $S \in \mathcal{E}$. It is said to be a "ternary" solution, otherwise.

A solution d is dictatorial if there exists $j \in E$ such that $d_S(x, y) = 1$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$. d is nondictatorial if it is not dictatorial.

An ASWF on Ω, f, and a solution to an IP on the same Ω, d, are said to correspond if, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

3 Arrovian social welfare functions and integer programming

The first formulation of an IP on Ω was proposed by Sethuraman et al. (2003), for the case where $d_S(x, y) \in \{0, 1\}$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$. This binary IP, which we will call IP0, consists of the following set of constraints:

$$d_E(x, y) = 1, \qquad\qquad (i)$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \qquad\qquad (ii)$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \qquad (iii)$$

for all triples of alternatives x, y, z and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy

the following conditions (hereafter referred to as Conditions $(*)$):

$$A \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}z\mathbf{p}y,$$

$$B \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}x\mathbf{p}z,$$

$$C \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}y\mathbf{p}x,$$

$$U \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}y\mathbf{p}z,$$

$$V \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}x\mathbf{p}y,$$

$$W \neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}z\mathbf{p}x.$$

By introducing integer programming, Sethuraman et al. (2003) were able to provide a new representation of ASWFs with respect to the axiomatic one previously used in the Arrow's tradition. In particular they showed, in their Theorem 1, that there exists a one-to-one correspondence between the set of the solutions to IP0 on Ω and the set of the ASWFs without ties on the same Ω. Sethuraman et al. (2003) also built up a second binary IP on Ω, for many respects related to the work of Kalai and Muller (1977) on nondictatorial ASWFs.

Kalai and Muller (1977) introduced the following condition of decomposability to characterize the domains of antisymmetric preference orderings admitting nondictatorial ASWFs without ties.

Ω is said to be decomposable (henceforth, KM decomposable) if there exists a set R, with $TR \subsetneq R \subsetneq \mathcal{A}^2$, satisfying the following conditions.

Condition I. For every two pairs $(x, y), (x, z) \in NTR$, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition II. For every two pairs $(x, y), (x, z) \in NTR$, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(z, x) \in R$ implies that $(y, x) \in R$.

Condition III. For every two pairs $(x, y), (x, z) \in NTR$, if there exists $\mathbf{p} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

Condition IV. For every two pairs $(x, y), (x, z) \in NTR$, if there exists $\mathbf{p} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$, then $(z, x) \in R$ implies that $(y, x) \in R$ or $(z, y) \in R$.

In the second IP introduced by Sethuraman et al. (2003), which we will call IP0$'$, constraint (iii) is replaced by the following set of constraints:

$$d_S(x, y) \leq d_S(x, z), \tag{iv}$$

$$d_S(z, x) \leq d_S(y, x), \tag{v}$$

for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x, y) + d_S(y, z) \leq 1 + d_S(x, z), \qquad\qquad (vi)$$

$$d_S(z, y) + d_S(y, x) \geq d_S(z, x), \qquad\qquad (vii)$$

for all triples x, y, z such that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and for all $S \in \mathcal{E}$.

Constraints (iv) and (v) translate, in terms of variables $d_S(x, y)$, Conditions I and II of Kalai and Muller (1977). In their Claim 1, Sethuraman et al. (2003) showed that these constraints are special cases of (iii). Constraints (vi) and (vii) translate Conditions III and IV of Kalai and Muller (1977). In their Claim 2, Sethuraman et al. (2003) showed that also these constraints are special cases of (iii). Their analysis established that any solution d to IP0 on Ω is a solution to IP0$'$ on the same domain and that IP0 and IP0$'$ are equivalent in the case where $n = 2$.

We now prove that the set of constraints (iv)-(vii) exhibits problems of logical dependence. More precisely, the following proposition shows that one of the constraints (iv) and (v) is redundant.

Proposition 1. *d satisfies (i), (ii), and (iv) if and only if it satisfies (i), (ii), and (v).*

Proof. Suppose that d satisfies (i), (ii), and (iv). Consider a triple x, y, z. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(z, x) > d_S(y, x),$$

for some $S \in \mathcal{E}$. Then, $d_S(z, x) = 1$, $d_S(y, x) = 0$. But then, $d_{S^c}(x, z) = 0$, $d_{S^c}(x, y) = 1$. This implies that

$$d_{S^c}(x, y) > d_{S^c}(x, z),$$

contradicting (iv). Therefore, d satisfies (i), (ii), and (v). Suppose that d satisfies (i), (ii), and (v). Consider a triple x, y, z. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(x, y) > d_S(x, z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x, y) = 1$, $d_S(x, z) = 0$. But then, $d_{S^c}(y, x) = 0$, $d_{S^c}(z, x) = 1$. This implies that

$$d_{S^c}(z, x) > d_{S^c}(y, x),$$

contradicting (v). Therefore, d satisfies (i), (ii), and (iv). $\blacksquare$

Moreover, the following proposition shows that one of the constraints (vi) and (vii) is redundant.

Proposition 2. *d satisfies (i), (ii), and (vi) if and only if it satisfies (i), (ii), and (vii).*

Proof. Suppose that d satisfies (i), (ii), and (vi). Consider a triple x, y, z. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(z, y) + d_S(y, x) < d_S(z, x),$$

for some $S \in \mathcal{E}$. Thus, $d_S(z, y) = 0$, $d_S(y, x) = 0$, and $d_S(z, x) = 1$. But then, $d_{S^c}(y, z) = 1$, $d_{S^c}(x, y) = 1$, and $d_{S^c}(x, z) = 0$. This implies that

$$d_{S^c}(x, y) + d_{S^c}(y, z) > 1 + d_{S^c}(x, z),$$

contradicting (vi). Therefore, d satisfies (i), (ii), and (vii). Suppose that d satisfies (i), (ii), and (vii). Consider a triple x, y, z. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(x, y) + d_S(y, z) > 1 + d_S(x, z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x, y) = 1$, $d_S(y, z) = 1$, and $d_S(x, z) = 0$. But then, $d_{S^c}(y, x) = 0$, $d_{S^c}(z, y) = 0$, and $d_{S^c}(z, x) = 1$. This implies that

$$d_{S^c}(z, y) + d_{S^c}(y, x) < d_{S^c}(z, x),$$

contradicting (vii). Therefore, d satisfies (i), (ii), and (vi). ∎

We now introduce the generalization of IP0 to the case where $d_S(x, y) = \frac{1}{2}$, for some $(x, y) \in NTR$ and for some $S \in \mathcal{E}$, proposed by Busetto et al. (2015). This first IP on Ω proposed by Busetto et al. (2015), called IP1, consists of the following set of constraints:

$$d_E(x, y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

if $d_{A \cup U \cup V}(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup V \cup W}(z, x) \in \{0, 1\}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) = \frac{3}{2}, \qquad (4)$$

if $d_{A \cup U \cup V}(x, y) = \frac{1}{2}$ or $d_{B \cup U \cup W}(y, z) = \frac{1}{2}$ or $d_{C \cup V \cup W}(z, x) = \frac{1}{2}$, for all triples of alternatives x, y, z and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$.

Busetto et al. (2015) showed that this ternary program can be used to provide a general representation of ASWFs, with and without ties in the range. In particular they showed, in their Theorem 1, that there exists a one-to-one correspondence between the set of the solutions to IP1 on a given Ω and the set of all ASWFs on the same Ω. We now restate this fundamental theorem as it will be systematically used in the rest of the paper.

Theorem 1. *Consider a domain Ω. Given an ASWF on Ω, f, there exists a unique solution to IP1 on Ω, d, which corresponds to f. Given a solution to IP1 on Ω, d, there exists a unique ASWF on Ω, f, which corresponds to d.*

Proof. See the proof of Theorem 1 in Busetto et al. (2015). ∎

Busetto et al. (2015) also introduced a second ternary IP on Ω, which incorporates, like IP0$'$ proposed by Sethuraman et al. (2003), a reformulation of Conditions I-IV of Kalai and Muller (1977). In constructing it, they eliminated the redundancies inherent in IP0$'$, we have exhibited in Propositions 1 and 2. In fact, this second ternary IP, called IP2, consists of constraints (1), (2), and the following four logically independent constraints:

$$d_S(x, y) \leq d_S(x, z), \qquad (5)$$

if $d_S(x, y) \in \{0, 1\}$;

$$d_S(x, y) < d_S(x, z), \qquad (6)$$

if $d_S(x, y) = \frac{1}{2}$, for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x, y) + d_S(y, z) \leq 1 + d_S(x, z), \qquad (7)$$

if $d_S(x, y), d_S(y, z) \in \{0, 1\}$;

$$d_S(x, y) + d_S(y, z) = \frac{1}{2} + d_S(x, z), \qquad (8)$$

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if $d_S(x, y) = \frac{1}{2}$ or $d_S(y, z) = \frac{1}{2}$, for all triples x, y, z such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, and for all $S \in \mathcal{E}$.

We now restate two propositions proved in Busetto et al. (2015) which establish the relationships between IP1 and IP2 and which we shall systematically use in the rest of the paper.

Proposition 3. *If d is a solution to IP1 on Ω, then it is a solution to IP2 on the same Ω.*

Proof. See the proof of Proposition 1 in Busetto et al. (2015). ∎

Proposition 4. *Let $n = 2$. If d is a solution to IP2 on Ω, then it is a solution to IP1 on the same Ω.*

Proof. See the proof of Proposition 2 in Busetto et al. (2015). ∎

We conclude this section restating the Arrow impossibility theorem as a corollary to the following impossibility result for nondictatorial solutions of IP1.

Theorem 2. *Let $\Omega = \Sigma$. If d is a solution to IP1 on Ω, then d is dictatorial.*

Proof. It follows by adapting, *mutatis mutandis*, the proof of Theorem 2 in Sethuraman et al. (2003) to IP1. ∎

Corollary 1. *Let $\Omega = \Sigma$. If f is an ASWF on Ω, then f is dictatorial.*

Proof. It is an immediate consequence of Theorems 1 and 2. ∎

4 Nondictatorial Arrovian social welfare functions and integer programming

In this section, we use the integer programs developed above to deal with the issues concerning the dictatorship property of ASWFs. To begin with, we focus here on ASWFs without ties.

Kalai and Muller (1977) were the first who provided a complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs without ties. They did this by means of two theorems. In their Theorem 1, they showed that, for a given domain Ω, there exists a nondictatorial ASWF without ties for $n = 2$ if and only if, for the same Ω, there exists a nondictatorial ASWF without ties for $n > 2$. In their Theorem 2, they showed that there exists a nondictatorial ASWF

without ties on Ω for $n \geq 2$ if and only if Ω satisfies the conditions of KM decomposability introduced in Section 3.

Sethuraman et al. (2003) opened the way to an analysis of the problem of dictatorship in terms of integer programming. More precisely, they showed, in their Theorem 8, a result establishing a one-to-one correspondence between the nondictatorial binary solutions of IP0 for $n = 2$ and its nondictatorial binary solutions for $n > 2$. Their result can be restated in terms of IP1 as follows.

Theorem 3. *There exists a nondictatorial binary solution to IP1 on Ω, d, for $n = 2$, if and only if there exists a nondictatorial binary solution to IP1 on Ω, d^*, for $n > 2$.*

Proof. It follows by adapting, *mutatis mutandis*, the proof of Theorem 8 in Sethuraman et al. (2003) to IP1. ∎

Theorem 1 in Kalai and Muller (1977) can therefore be obtained as a corollary of Theorem 1.

Corollary 2. *There exists a nondictatorial ASWF without ties on Ω, f, for $n = 2$, if and only if there exists a nondictatorial ASWF without ties on Ω, f^*, for $n > 2$.*

Proof. It is an immediate consequence of Theorems 1 and 3. ∎

Now, we go forward along the line opened by Sethuraman et al. (2003), providing a characterization of domains admitting nondictatorial binary solutions to IP1. As it will be made clear shortly, this result is the heart of a new, simpler proof of Theorem 2 in Kalai and Muller (1977) for nondictatorial ASWFs without ties, in terms of integer programming.

In order to obtain our characterization theorem, we need to use the reformulation of the concept of KM decomposability suitable to be applied within the analytical context of IP1. This reformulation is based on the existence of two sets, $R_1, R_2 \in \mathcal{A}^2$, instead of only one, which satisfy the two conditions we are going to introduce.

Consider a set $R \subset \mathcal{A}^2$. Consider the following conditions on R.

Condition 1. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition 2. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be decomposable if and only if there exist two sets R_1 and R_2, with $\emptyset \subsetneq R_i \subsetneq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2.

With regard to this definition of a decomposable domain, let us remind the main differences with the original notion of KM decomposability already noticed by Busetto et al. (2015). Conditions 1 and 2 differ from the corresponding Conditions I and III as the former refer to triples, rather than pairs, of alternatives. Moreover, Condition 2 is reformulated in terms of a pair of preference orderings, instead of only one. This is consistent with the formulation of our constraints (7) and (8), which are in fact a reinterpretation of Condition 2 in terms of integer programming. Also, this new definition of a decomposable domain does not require that R_1 and R_2 contain TR, whereas KM decomposability requires that R contains TR and it requires that R_1 and R_2 satisfy only two conditions, instead of four, as required by KM decomposability. As Proposition 5 below makes it clear, this implies a redundancy of Conditions II and IV of KM decomposability, which parallels the redundancy of constraints (v) and (vii) proved in Propositions 1 and 2.

On the basis of the reformulation of the concept of decomposability, we now state and prove the characterization theorem.

Theorem 4. *There exists a nondictatorial binary solution to IP2 on Ω, d, for $n = 2$, if and only if Ω is decomposable.*

Proof. Let d be a nondictatorial binary solution to IP2 on Ω, for $n = 2$. Let $R_1 = \{(x, y) \in NTR : d_{\{1\}}(x, y) = 1\}$ and $R_2 = \{(x, y) \in NTR : d_{\{2\}}(x, y) = 1\}$. Then, for all $(x, y) \in NTR$, $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$, as d satisfies (2). Moreover, $\emptyset \subsetneq R_i \subsetneq NTR$, $i = 1, 2$, as d is nondictatorial. Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x, y) \in R_1$ and $(x, z) \notin R_1$ Then, $d_{\{1\}}(x, y) = 1$ and

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z),$$

contradicting (5). Hence, R_i, $i = 1, 2$, satisfies Condition 1. Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x, y), (y, z) \in R_1$, and $(x, z) \notin R_1$. Then, $d_{\{1\}}(x, y) = 1$, $d_{\{1\}}(y, z) = 1$, and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z),$$

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contradicting (7). Hence, R_i, $i = 1, 2$, satisfies Condition 2. We have proved that Ω is decomposable. Conversely, suppose that Ω is decomposable. Then, there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2. Determine d as follows. For each $(x, y) \in NTR$, let $d_\emptyset(x, y) = 0$, $d_E(x, y) = 1$; moreover, let $d_{\{i\}}(x, y) = 1$ if and only if $(x, y) \in R_i$; $d_{\{i\}}(x, y) = 0$ if and only if $(x, y) \notin R_i$, for $i = 1, 2$. Then, d satisfies (1) and (2) as, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$. Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z).$$

Then, we have $(x, y) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 1. Therefore, d satisfies (5). Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z).$$

Then, we have $(x, y), (y, z) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 2. Therefore, d satisfies (7). d is nondictatorial as $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$. Hence, d is a nondictatorial binary solution to IP2 on Ω. ∎

The previous result provides a simplified proof of Theorem 2 in Kalai and Muller (1977) since this theorem can be obtained as a corollary of Theorem 4.

Corollary 3. *There exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$, if and only if Ω is decomposable.*

Proof. It is a consequence of Theorems 1 and 4, Propositions 3 and 4, and Corollary 2. ∎

From the previous corollary, we obtain a result, which, as anticipated above, establishes the equivalence between the new notion of decomposability and KM decomposability, and implies that Conditions II and IV are redundant.[1]

Proposition 5. *Ω is KM decomposable if and only if it is decomposable.*

Proof. It is an immediate consequence of Theorem 2 in Kalai and Muller (1977) and Corollary 3. ∎

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[1]Busetto et al. (2015), in their Proposition 3, provided a direct proof of this result.

Busetto et al. (2015) showed, in their Theorem 3, a result establishing a one-to-one correspondence relation between the ternary solutions of IP1 for $n = 2$ and its ternary solutions for $n > 2$. We now restate their result.

Theorem 5. *There exists a nondictatorial ternary solution to IP1 on Ω, d, for $n = 2$, if and only if there exists a nondictatorial ternary solution to IP1 on Ω, d^*, for $n > 2$.*

Proof. See the proof of Theorem 3 in Busetto et al. (2015). ∎

From Theorem 5, Busetto et al. (2015) obtained the following corollary, which extends Theorem 1 in Kalai and Muller (1977) to the case of nondictatorial ASWFs with ties.

Corollary 4. *There exists a nondictatorial ASWF with ties on Ω, f, for $n = 2$, if and only if there exists a nondictatorial ASWF with ties on Ω, f^*, for $n > 2$.*

Proof. It is an immediate consequence of Theorems 1 and 5. ∎

In order to obtain their characterization theorem for nondictatorial ASWFs with ties, Busetto et al. (2015) needed to restrict further the condition of decomposability, introducing a new notion which they defined as strict decomposability. We now provide the notion of strict decomposability.

Given a set $R \subset \mathcal{A}^2$, consider the following conditions on R.

Condition 3. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R^*$ implies that $(x, z) \in R$.

Condition 4. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples of alternatives x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R^*$ imply that $(x, z) \in R$, and $(x, y) \in R^*$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be strictly decomposable if and only if there exist four sets R_1, R_2, R_1^*, and R_2^*, with $R_i \subsetneq NTR$, $\emptyset \subsetneq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, R_i, $i = 1, 2$, satisfies Condition 1; R_i and R_i^*, $i = 1, 2$, satisfy Condition 2; each pair (R_i, R_i^*), $i = 1, 2$, satisfies Conditions 3 and 4.

On the basis of the notion of strict decomposability, Busetto et al. (2015), in their Theorem 4, provided the following characterization of domains admitting nondictatorial ternary solutions to IP1.

Theorem 6. *There exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$, if and only if Ω is strictly decomposable.*

Proof. See the proof of Theorem 4 in Busetto et al. (2015). ∎

Busetto et al. (2015) then proved, in their Theorem 5, the following generalization of Theorem 2 in Kalai and Muller (1977) for nondictatorial ASWFs without ties, which we restate as a corollary.

Corollary 5. *There exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$, if and only if Ω is strictly decomposable.*

Proof. It is a straightforward consequence of Theorems 1 and 6, Propositions 3 and 4, and Corollary 4. ∎

The following proposition restates Theorem 7 in Busetto et al. (2015) which shows that a strictly decomposable domain is always decomposable.

Proposition 6. *If a domain Ω is strictly decomposable, then it is decomposable.*

Proof. See the proof of Theorem 7 in Busetto et al. (2015). ∎

Example 2 in Busetto et al. (2015) shows that the converse of Proposition 6 does not hold.

We now consider a further investigation on domains admitting nondictatorial ASWFs which was initiated by Kalai and Ritz (1978). In particular, they studied the relationship between their notion of a domain containing an inseparable ordered pair and the notion of a KM decomposable domain.

According to Kalai and Ritz (1978), Ω is said to contain an inseparable ordered pair if there exists $(u, v) \in NTR$ such that, for no $\mathbf{p} \in \Omega$ and $t \in \mathcal{A}$, $u\mathbf{p}t\mathbf{p}v$.

We will now extend the analysis to the relationship between the notion of a domain containing an inseparable ordered pair and the notion of a strictly decomposable domain introduced by Busetto et al. (2015). On the basis of IP2, we now state and prove the following result.

Theorem 7. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$.*

Proof. Suppose that Ω contains an inseparable ordered pair $(u, v) \in NTR$. Determine d as follows. For each $(x, y) \in NTR$, let $d_{\emptyset}(x, y) = 0$, $d_E(x, y) = 1$. Moreover, let $d_{\{1\}}(x, y) = 1$ and $d_{\{2\}}(y, x) = 0$, if and only if $(x, y) \neq (u, v)$; $d_{\{1\}}(x, y) = \frac{1}{2}$ and $d_{\{2\}}(y, x) = \frac{1}{2}$, if and only if $(x, y) = (u, v)$. Then,

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d satisfies (1) and (2). Consider a triple x, y, z. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y) \in \{0, 1\}$ and

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z).$$

Then, $(x, z) = (u, v)$. But then, (u, v) is not inseparable as $u\mathbf{p}y\mathbf{p}v$, a contradiction. Now, suppose that $d_{\{2\}}(x, y) \in \{0, 1\}$ and

$$d_{\{2\}}(x, y) > d_{\{2\}}(x, z).$$

Then, we have $d_{\{2\}}(x, y) = 1$, a contradiction. Therefore, d satisfies (5). Suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z).$$

Then, we have $(x, y) = (u, v)$. But then, we have $d_{\{1\}}(x, z) = 1$, a contradiction. Suppose that $d_{\{2\}}(x, y) = \frac{1}{2}$ and

$$d_{\{2\}}(x, y) \geq d_{\{2\}}(x, z).$$

Then, we have $(x, y) = (v, u)$. But then, (u, v) is not inseparable as $u\mathbf{p}z\mathbf{p}v$, a contradiction. Therefore, d satisfies (6). Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y), d_{\{1\}}(y, z) \in \{0, 1\}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z).$$

Then, we have $(x, z) = (u, v)$. But then, (u, v) is not inseparable as $u\mathbf{p}y\mathbf{p}v$, a contradiction. Now, suppose that $d_{\{2\}}(x, y), d_{\{2\}}(y, z) \in \{0, 1\}$ and

$$d_{\{2\}}(x, y) + d_{\{2\}}(y, z) > 1 + d_{\{2\}}(x, z).$$

Then, we have $d_{\{2\}}(x, y) = 1$ and $d_{\{2\}}(y, z) = 1$, a contradiction. Therefore, d satisfies (7). Suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z).$$

Then, we have $d_{\{1\}}(x, z) = 0$, a contradiction. Suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) < \frac{1}{2} + d_{\{1\}}(x, z).$$

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Then, we have $d_{\{1\}}(y,z) = 0$, a contradiction. Suppose that $d_{\{2\}}(x,y) = \frac{1}{2}$ and

$$d_{\{2\}}(x,y) + d_{\{2\}}(y,z) > \frac{1}{2} + d_{\{2\}}(x,z).$$

Then, we have $d_{\{2\}}(y,z) = 1$, a contradiction. Suppose that $d_{\{2\}}(x,y) = \frac{1}{2}$ and

$$d_{\{2\}}(x,y) + d_{\{2\}}(y,z) < \frac{1}{2} + d_{\{2\}}(x,z).$$

Then, we have $d_{\{2\}}(x,z) = 1$, a contradiction. Therefore, d satisfies (8). Hence, d is a nondictatorial ternary solution to IP2 on Ω. ∎

We can then obtain a corollary of Theorem 7, establishing that a domain Ω which contains an inseparable ordered pair always admits a nondictatorial ASWF with ties, for $n \geq 2$.

Corollary 6. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$.*

Proof. It is a straightforward consequence of Theorems 1 and 7, Proposition 4 and Corollary 4. ∎

Theorem 6 states that there exists a nondictatorial ternary solution to IP2 on Ω if and only if Ω is strictly decomposable. By exploiting this result, we establish here the relationship between the notions of a domain containing an inseparable ordered pair and of a strictly decomposable domain.

Proposition 7. *If Ω contains an inseparable ordered pair, then it is strictly decomposable.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, there exists a nondictatorial ternary solution to IP2 on Ω, d, for $n = 2$, by Theorem 7. Bur then, Ω is strictly decomposable, by Theorem 6. ∎

The following example shows that the converse of Proposition 7 does not hold.

Example 1. *Let A be the closed interval $[0,1]$ of the real line and $\Omega = \{\mathbf{p}, \mathbf{p}^{-1}\}$, where $\mathbf{p}$ is such that, if $x, y \in [0,1]$ and $x > y$, then $x\mathbf{p}y$. Then, Ω is strictly decomposable but it does not contain an inseparable ordered pair.*

Proof. Let $V_i = \emptyset$, $i = 1, 2$, $V_1^* = \{(x,y) \in NTR : x\mathbf{p}y\}$, $V_2^* = \{(x,y) \in NTR : x\mathbf{p}^{-1}y\}$. Then, we have $\emptyset \subsetneq V_i^* \subset NTR$, $i = 1, 2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in V_1^*$ if and only if $(y,x) \in V_2^*$. Finally, V_i^*, $i =$

1, 2, satisfies Condition 2. Therefore, Ω is strictly decomposable. Moreover, it is straightforward to verify that Ω does not contain an inseparable ordered pair. ∎

The main result in Kalai and Ritz (1978), establishing the relationship between the notions of a domain containing an inseparable ordered pair and of a decomposable domain, can now be straightforwardly obtained from Theorem 7.

Proposition 8. *If Ω contains an inseparable ordered pair, then it is decomposable.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, it is strictly decomposable, by Proposition 7. But then, it is decomposable, by Proposition 6. ∎

The converse of Proposition 8 does not hold. This is an immediate implication of Example 2 in Busetto et al. (2015) that exhibits a decomposable domain which does not contain an inseparable ordered pair. Instead, Proposition 8 has the following implication, which concerns the existence of nondictatorial ASWFs without ties on domains containing an inseparable ordered pair.

Corollary 7. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, it is decomposable, by Proposition 8. But then, there exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$, by Corollary 3. ∎

The next result concludes the analysis, in terms of integer programming, of the relationships of the notion of a domain containing an inseparable ordered pair with those of a strictly decomposable domain, and a decomposable domain.

Proposition 9. *If Ω is decomposable but not strictly decomposable, then it does not contain an inseparable ordered pair.*

Proof. It is a straightforward consequence of Proposition 7. ∎

5 Simple majority rule and integer programming

In this section, we use integer programming to determine the domains on which the Simple Majority Rule (SMR) is a nondictatorial ASWF and we

compare them with the domains admitting nondictatorial ASWFs analyzed in the previous section. We start with some preliminary definition.

A solution d to an IP on Ω is a SMR solution if for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $d_S(x, y) = 1$ if and only if $|S| > |S^c|$, $d_S(x, y) = \frac{1}{2}$ if and only if $|S| = |S^c|$, and $d_S(x, y) = 0$ if and only if $|S| < |S^c|$. It is immediate to verify that a SMR solution to an IP on Ω, d, is binary if and only if n is odd and ternary if and only if n is even.

An ASWF on Ω, f, is said to be based on the SMR if it corresponds to a solution to IP1 on the same Ω, d, which is a SMR solution. It is immediate to verify that an ASWF on Ω, f, based on the SMR is nondictatorial without ties if and only if n is odd and nondictatorial with ties if and only if n is even.

We now restate a theorem, proved by Sethuraman et al. (2003), which is an integer programming version of a result showed by Sen (1966). The result is based on the following domain restriction.

A domain Ω is said to contain a Condorcet triple if there are triple x, y, z and $\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3 \in \Omega$ such that $x\mathbf{p}^1 y\mathbf{p}^1 z$, $y\mathbf{p}^2 z\mathbf{p}^2 x$, and $z\mathbf{p}^3 x\mathbf{p}^3 y$.

We can now restate Theorem 5 in Sethuraman et al. (2003)

Theorem 8. *Let n be odd. There exists a SMR binary solution to IP1 on Ω, d, if and only if Ω does not contain a Condorcet triple.*

Proof. It follows by adapting, *mutatis mutandis*, the proof of Theorem 5 in Sethuraman et al. (2003) to IP1. ∎

We can then easily derive Theorem 1 in Sen (1966) as a corollary to Theorem 8.

Corollary 8. *Let n be odd. There exists an ASWF on Ω, f, based on the SMR if and only if Ω does not contain a Condorcet triple.*

Proof. It follows from Theorems 1 and 8. ∎

In view to overcome his impossibility theorem, Arrow (1963) extended the notion of a single-peaked domain previously introduced by Black (1948). In particular, he proposed the following definition of a single-peaked domain.

Given $\mathbf{q} \in \Sigma$, a domain Ω is said to be single-peaked relative to $\mathbf{q}$ if, for all triples x, y, z and for each $\mathbf{p} \in \Omega$, $x\mathbf{q}y\mathbf{q}z$ and $x\mathbf{p}y$ implies that $y\mathbf{p}z$.

We shall now show that a single-peaked domain admits a SMR binary solution to IP1 when n is odd.

Theorem 9. *Let n be odd. If Ω is single-peaked relative to $\mathbf{q}$, then there exists a SMR binary solution to IP1 on Ω, d.*

Proof. Suppose that Ω is single-peaked relative to $\mathbf{q}$. Determine d as follows. For each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $d_S(x, y) = 1$ if and only if $|S| > |S^c|$, and $d_S(x, y) = 0$ if and only if $|S| < |S^c|$. Then, it is straightforward to verify that d satisfies (1) and (2). Consider a triple x, y, z and suppose, without loss of generality, that $x\mathbf{q}y\mathbf{q}z$. Then, we have that $A = \emptyset$ and $V = \emptyset$ as Ω is single-peaked relative to $\mathbf{q}$ and $d_U(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup W}(z, x) \in \{0, 1\}$ as n is odd. Suppose that

$$d_U(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup W}(z, x) > 2.$$

Then, we have that $d_U(x, y) = 1$, $d_{B \cup U \cup W}(y, z) = 1$, and $d_{C \cup W}(z, x) = 1$. But then, we must have that $|B| + |C| + |U| + |W| = n$ as $A = \emptyset$ and $V = \emptyset$ and $|B| + |C| + |U| + |W| > n$ as $|B| + |U| + |W| > \frac{n}{2}$ and $|C| + |W| > \frac{n}{2}$, a contradiction. Therefore, d satisfies (3). Hence, d is a SMR binary solution to IP1 on Ω. ∎

The Arrow possibility theorem then follows as a corollary of Theorem 9.

Corollary 9. *Let n be odd. If Ω is single-peaked relative to $\mathbf{q}$, then there exists an ASWF on Ω, f, based on the SMR.*

Proof. It follows from Theorems 1 and 9. ∎

Theorems 8 and 9 hold when n is odd. We shall now use integer programming to state and prove a theorem, first showed by Inada (1969), which provides a SMR solution to IP1 for any n. The result is based on the following domain restriction.

A domain Ω is said to be echoic if, for all triples x, y, z and for each $\mathbf{p}, \mathbf{q} \in \Omega$, $x\mathbf{p}y\mathbf{p}z$ implies that $x\mathbf{q}z$.[2]

We shall now show that an echoic domain admits a SMR solution to IP1 for any n.

Theorem 10. *In Ω is echoic, then there exists a SMR solution to IP1 on Ω, d.*

Proof. Suppose that Ω is echoic. Determine d as follows. For each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $d_S(x, y) = 1$ if and only if $|S| > |S^c|$, $d_S(x, y) = \frac{1}{2}$ if and only if $|S| = |S^c|$, and $d_S(x, y) = 0$ if and only if $|S| < |S^c|$. Then, it is straightforward to verify that d satisfies (1) and (2). Consider a triple x, y, z and suppose, without loss of generality, that $\mathbf{p} \in \Omega$ and $x\mathbf{p}y\mathbf{p}z$. Suppose first

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[2]This definition is based on the definition of echoic preferences provided by Condition $(B)'$ in Inada (1969).

that $\mathbf{q} \in \Omega$ with $x\mathbf{q}z\mathbf{q}y$. Then, we have that $B = \emptyset$, $C = \emptyset$, $V = \emptyset$, $W = \emptyset$. But then, we also have that $d_{A \cup U}(x,y) = d_E(x,y) = 1$ and $d_\emptyset(z,x) = 0$. Consider the case where n is odd. Then, we have that $d_U(y,z) \in \{0,1\}$ as n is odd. But then, it must be that

$$d_{A \cup U}(x,y) + d_U(y,z) + d_\emptyset(z,x) \leq 2,$$

as $d_{A \cup U}(x,y) = d_E(x,y) = 1$ and $d_\emptyset(z,x) = 0$. Therefore, d satisfies (3). Consider the case where n is even. Then, we have that $d_U(y,z) \in \{0, \frac{1}{2}, 1\}$. Suppose that $d_U(y,z) = 0$ or $d_U(y,z) = 1$. Then, we must have that

$$d_{A \cup U}(x,y) + d_U(y,z) + d_\emptyset(z,x) \leq 2,$$

as $d_{A \cup U}(x,y) = d_E(x,y) = 1$ and $d_\emptyset(z,x) = 0$. Therefore, d satisfies (3). Suppose that $d_U(y,z) = \frac{1}{2}$. Then, it must be that

$$d_{A \cup U}(x,y) + d_U(y,z) + d_\emptyset(z,x) = \frac{3}{2},$$

as $d_{A \cup U}(x,y) = d_E(x,y) = 1$ and $d_\emptyset(z,x) = 0$. Therefore, d satisfies (4). Suppose now that $\mathbf{q} \in \Omega$ with $y\mathbf{q}x\mathbf{q}z$. Then, by using *mutatis mutandis* the above argument, it follows that d satisfies (3) or (4). Hence, d is a SMR solution to IP1 on Ω. ∎

We can then easily derive Theorem $2'$ in Inada (1969) as a corollary to Theorem 10.

Corollary 10. *If Ω is echoic, then there exists an ASWF on Ω, f, based on the SMR.*

Proof. It follows from Theorems 1 and 10. ∎

We now investigate the relationships among the domains admitting non-dictatorial ASWFs and those admitting an ASWF based on the SMR. We first consider the relationship between a domain which does not contain a Condorcet triple and a decomposable domain.

Proposition 10. *If Ω does not contain a Condorcet triple, then it is decomposable.*

Proof. Suppose that Ω does not contain a Condorcet triple. Let n be odd. Then, there exists a SMR binary solution to IP1 on Ω, d, by Theorem 8. But then, there exists a nondictatorial binary solution to IP1 on Ω, d^*, for $n = 2$, by Theorem 3, which is also a solution to IP2 on the same Ω, by Proposition 3. Hence, Ω is decomposable, by Theorem 4. ∎

We can now provide an example, borrowed from Example 2 in Busetto et al. (2015), which shows that a domain Ω which does not contain a Condorcet triple is not necessarily strictly decomposable.

Example 2. *Let $A = \{a, b, c, d\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c\mathbf{p}d, c\mathbf{p}a\mathbf{p}d\mathbf{p}b,$ $d\mathbf{p}c\mathbf{p}b\mathbf{p}a, b\mathbf{p}d\mathbf{p}a\mathbf{p}c\}$. Then, Ω does not contain a Condorcet triple but it is not strictly decomposable.*

Proof. It is straightforward to verify that Ω does not contain a Condorcet triple. Nevertheless, by the proof of Example 2 in Busetto et al. (2015), it is not strictly decomposable. ∎

Example 2 shows that the following proposition holds non-vacuously.

Proposition 11. *If Ω does not contain a Condorcet triple and it is not strictly decomposable, then it does not contain an inseparable ordered pair.*

Proof. It is a straightforward consequence of Proposition 7. ∎

In their Example 2, Kalai and Muller (1977) showed that, for any $\mathbf{q} \in \Sigma$, if Ω single-peaked relative to $\mathbf{q}$, then it is KM decomposable. We now provide a direct proof of this result in our framework.

Proposition 12. *If Ω is single-peaked relative to $\mathbf{q}$, then it is decomposable.*

Proof. Suppose that Ω is single-peaked relative to $\mathbf{q}$. Let n be odd. Then, there exists a SMR binary solution to IP1 Ω, d, by Theorem 9. But then, there exists a nondictatorial binary solution to IP1 on Ω, d^*, for $n = 2$, by Theorem 3, which is also a solution to IP2 on the same Ω, by Proposition 3. Hence, Ω is decomposable, by Theorem 4. ∎

Our last proposition shows that an echoic domain is strictly decomposable.

Proposition 13. *In Ω is echoic, then it is strictly decomposable.*

Proof. Suppose that Ω echoic. Let $n = 2$. Then, there exists a SMR ternary solution to IP1 Ω, d, by Theorem 10. But then, d is a ternary solution to IP2 on Ω, by Proposition 3. Hence Ω is strictly decomposable, by Theorem 6. ∎

The following example shows that the converse of Propositions 10, 12, and 13 does not hold.

Example 3. *Let $\mathcal{A} = \{a, b, c\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}c\mathbf{p}b, b\mathbf{p}a\mathbf{p}c, b\mathbf{p}c\mathbf{p}a, c\mathbf{p}a\mathbf{p}b,$ $c\mathbf{p}b\mathbf{p}a\}$. Then, Ω is strictly decomposable and decomposable but it contains*

a Condorcet triple, it is not single peaked relative to any $\mathbf{q} \in \Sigma$, and it is not echoic.

Proof. It is immediate to verify that the ordered pair (a, c) is inseparable. Then, Ω is strictly decomposable, by Proposition 7 and decomposable, by Proposition 8. Nevertheless, Ω contains a Condorcet triple, it is not single peaked, and it is not echoic as it contains five preference profiles. ∎

6 Conclusion

In this paper, we have systematically used integer programming to characterize the domains admitting nondictatorial ASWFs with and without ties. Moreover, we have established some relationships among those domains, which should clarify the Arrovian foundations of mechanism design, the "genuflection," in the framework inspired by Vohra (2011). We have also revised some results on the SMR, which is a classical example of a nondictatorial ASWF with and without ties. In our analysis we have assumed, as Sethuraman et al. (2003) did in their main analysis, the common preference domain framework, in the language of Le Breton and Weymark (1996). We leave for further analysis the extension to the case in which agents' domains are different which would require that we recast the analysis of nondictatorial ASWFs and of the SMR in the framework of the generalized IP sketched by Sethuraman et al. (2003) to deal with this case.

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