Cheap Talk Advertising in Auctions: Horizontally vs Vertically Differentiated Products

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Cheap Talk Advertising in Auctions: Horizontally vs Vertically Differentiated Products[*]

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Abstract

This paper explores the possibilities for sellers to usefully transmit product information to buyers by cheap talk public advertising. We explore two polar cases, contrasting vertically differentiated products (à la Milgrom Weber's (1982) general symmetric model) with horizontally differentiated products (à la Hotelling's (1929) line). We consider both the message only case and where reserve price-message pairs can be chosen by the seller. For horizontally differentiated products partitional message-only informative equilibria are shown to exist providing the number of bidders is sufficiently large. The equilibrium is characterized by more precise information provided for less popular product attributes. The seller optimal disclosure policy displays a complementarity relationship between the number of bidders and the amount of product information disclosed. In contrast, for the vertically differentiated products benchmark, message-only informative equilibria do not exist. With reserve prices, informative equilibria exist in both cases. For the vertical case these equilibria yield lower seller revenue than uninformative equilibria. In the horizontal case with sufficiently large number of bidders higher revenue is possible and full disclosure becomes feasible and seller optimal in the limit.

Keywords: Cheap Talk; Information Disclosure; Auction; Horizontal Differentiation; Vertical Differentiation; Informative Equilibrium

JEL Classification: D44, D82, D83, L10, M37

1 Introduction

One of the important contributions in Milgrom and Weber's (1982) (hereinafter MW) seminal paper was to the question of optimal information disclosure. MW showed that in context of their general symmetric model the ex-ante expected revenue maximizing policy is to inform buyers as fully as possible. The object of choice of the seller here is the ex-ante disclosure *rule* which maps the information which the seller receives to the message transmitted to buyers. Although this modeling choice fits many economic scenarios, there are also certainly others in which it is more natural to think of sellers not as choosing a disclosure rule *before* they receive their private information, but rather simply trying to sell what they have. It also appears that in much product advertising the seller seeks to convey information about the product by vague statements or by imagery which give no right of legal redress[1]. In other words, much advertising is cheap-talk in which off the equilibrium path the informed seller will opportunistically send misleading messages if it is in her interest to do so.

In general, what information may be transmitted by cheap-talk depends on characteristics of the joint distribution of valuations of bidders, their private information and the information available to the seller. This paper explores two important polar cases representing the benchmark case of vertically differentiated (VD) products à la MW's (1982) general symmetric model and horizontal differentiated (HD) products à la Hotelling's (1929) line.

To be specific, in this paper, a seller offers a product to a number of *ex ante* homogenous bidders in a first price auction[2]. We focus on disclosure policies in the form of partitions over seller information, as is familiar in the literature on cheap-talk (Crawford Sobel (1982)). Defining the expected auction revenue conditional on a message as the interim expected revenue, it follows that since under cheap-talk, the seller is not committed to executing a preset disclosure rule, all the equilibrium interim expected revenues must be equal—otherwise the seller will deviate to report the message that generates the highest level of expected revenue, regardless of what the real attribute is.

In the benchmark case of VD products, it follows easily from the analysis of MW (1982) that a message-only informative equilibrium does not exist. This is because bidders share the same direction of preference for VD products, and the corresponding messages can also be ranked in terms of their induced interim expected revenues. As a result, the seller will always announce the message that generates the highest expected revenue, no

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[1]For instance, following a complaint by the FTC of misleading advertising for failure to mention high sodium content Campbell Soup Co. reverted from a "Soup is good food" campaign and returned to its "M'm m'm good" theme, Abernethy and Franke (1998). Intuitively, one does not expect to see equilibrium cheap talk adverts of the kind "M'm m'm good" when the alternative is "M'm m'm bad". But it seems more plausible to see an image of a child uttering "M'm m'm good" when the alternative is an image of a hungry worker saying "M'm m'm good". As Nelson (1974) notes, "advertisements for experience goods are dominantly *soft* or indirect information". This contrasts with *hard* information disclosure Milgrom (1981) and Grossman (1981) in which the seller must state the truth, but not necessarily the whole truth. And Celik (2014) considers this sort of information disclosure in a model of horizontally differentiated products.

[2]Our assumptions will mean that the revenue equivalence theorem holds so our results will hold for any standard auction.

matter what the real attribute is, and the revealed information is hence not credible. In contrast, a fully revealing equilibrium *can* be sustained in a message-price informative equilibrium, where setting reserve price message pairs is allowed. However, we show that this fully revealing informative equilibrium results in lower interim revenue than would obtain from null information disclosure with optimal reserve price. Therefore, the seller optimal price-message equilibrium consists of a single reserve price and null message.

The case of HD products leads to a richer set of possibilities. In our model, the effect of a message about the product attribute on bidder valuations can be decomposed into two distinct effects. One, which is uniform across bidders, is the *risk reduction* effect. That is, providing more precise information reduces the risk of mismatch for all the bidders, and thus raises their valuations. The other is the idiosyncratic effect of *attribute revelation*, if a message shifts the conditional expectation of the product attribute towards the extremes of the Hotelling line then given our distributional assumptions the product becomes less popular and the distribution of conditional valuations becomes worse by first order stochastic dominance. We thereby show that, in message-only informative equilibria (without reserve prices) equilibrium partitions must have the property that more precise information is revealed for less popular product attributes. Only in this way can the risk reduction effect and attribute revelation effects be balanced to yield equal interim revenue for all messages.

This necessary condition begs the question of existence. The condition implies that in an equilibrium with a given partition degree[3], it must be possible to divide the Hotelling line into the given number of intervals each having positions and lengths which are somehow balanced with each other. Our second result supplies the necessary existence theorem. We prove that, for each partition degree J, a message-only informative equilibrium can always be supported by a partition of degree J, as long as the number of bidders is sufficiently large. Rather than the usual fixed-point approach, we apply an Intermediate Value Theorem defined on partially order sets. Furthermore, we also show that, given the number of bidders, n, there exists a maximum partition degree below which a message-only informative equilibrium can be supported, and that maximum partition degree is nondecreasing in n. And in the limit, when n approaches infinity, the equilibrium partition will converge to a partition where all the messages are of the same level of signal precision.

The intuition for these results is that, with increasing number of bidders, the difference in popularity across different product attributes converges to zero, and the differences in interim expected revenues are increasingly determined by the difference in signal precision. Therefore, as the popularity across different attributes become more and more equal it becomes easier for the seller to adjust the precision of signals to satisfy the equal revenue conditions. In the limit, when the number of bidders converge to infinity and all the product attributes are of the same level of popularity, the equilibrium partitions will naturally converge to one where all the messages are of the same signal precision.

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[3]For an interval partition, the degree of partition J is defined as the number of subintervals in the partition. Roughly speaking, a larger J corresponds to finer partition of the attribute interval, and thus, on average, more precise information revealed to the bidders.

Third, we establish a complementarity relationship between the number of bidders and the optimal amount of information measured by the degree of equilibrium partitions. That is, when the number of bidders increases, it is better for seller revenue to construct an informative equilibrium using higher degree partitions. The intuition here is similar to that behind the existence theorem. When the number of bidders increase, the difference in popularity across different product attributes vanishes, and it is easier to construct informative equilibria using more precise signals, which in turn increase the valuations of the bidders and thus the expected revenue of the auction. Hence, roughly speaking, as the number of bidders increase, not only are sellers able (in some equilibria) to transmit more information in terms of higher degree partitions, but it is in their interests to do so.

When setting reserve prices is allowed at the same time as disclosing product information, we further show that full revealing equilibrium is sustainable in a message-price informative equilibrium for HD products, as in the case of VD products. This is because the seller can adjust the reserve prices such that all the interim expected revenues be equal. However, in contrast to the case of VD products, for HD products, a full-revealing message-price informative equilibrium can result in higher revenue level than null information with optimal reserve price, as long as the number of bidders is sufficiently large.

The remainder of this paper is organized as follows. Section 2 is a short discussion of related literature. Section 3 sets up our model. Section 4 examines cheap-talk information disclosure for VD products. Section 5 focuses on cheap-talk information disclosure for HD products. Specifically, Section 5.1 investigates the properties of the interim expected revenue. Section 5.2 provides some interesting characterizations of message-only informative equilibria. Section 5.3 proves an existence theorem for message-only informative equilibria. Section 5.4 shows the complementarity property of the optimal disclosure policies. Section 5.5 studies the message-price informative equilibria under cheap-talk. Section 6 re-investigates the information disclosure problem for HD products, yet under truthful disclosure. Section 7 is a short conclusion.

2 Related Literature

This paper examines cheap talk information disclosure for HD and VD products taking as given that the sale will take place a standard auction mechanism. In two interesting recent contributions Balestrieri and Izmalkov (2016) and Koessler and Skreta (2016) consider the design of more general mechanisms for selling goods in a HD context. Balestrieri and Izmalkov (2016) derive the optimal selling mechanism for a monopolist who is privately informed about the attributes of a horizontally differentiated good. They show that a rather rich class of mechanisms can obtain depending on the precise details of the model. For instance, optimal mechanism may involve type-specific probabilistic allocations, which can also be implemented through private transmission of different information to different buyer types. Koessler and Skreta (2016) consider a single buyer whose willingness to pay depends on his privately-known taste and on product characteristics privately known by the seller. They explore general properties of the class of selling procedures that can arise

as an equilibrium of the game in which the seller chooses mechanisms conditional on her information. Such mechanisms include for instance bilateral cheap talk.

Chakraborty and Harbaugh (2010) consider informative multidimensional cheap talk in which an expert can make credible comparative statements that trade off her incentive to exaggerate on each dimension. They give an example applied to product advertising in which the product in question has elements of both horizontal and vertical differentiation. This multidimensional trade-off theme which is developed further in Chakraborty and Harbaugh (2014) has some similarities with the mechanism supporting cheap talk equilibria in our model.

There is a strand of literature that studies information policies in various situations, such as in monopoly pricing (Lewis and Sappington, 1994; Johnson and Myatt, 2006), and in auctions (Board, 2009; Ganuza and Penalva, 2010; Hummel and McAfee, 2015). As in MW (1982), they assume that a seller commits to a preset information policy. However, rather than public disclosure of a signal, the information policy relates to the precision with which buyers will learn their valuations. Relatedly, but closer to our approach, Ganuza (2004) constructs an explicit HD model but in which buyer preferences are distributed on a circle, rather than line. In that paper the seller chooses a public disclosure rule à la MW (1982) (but from a restricted class of rules) rather than cheap talk. He shows that the optimal amount of information disclosure increases with the number of bidders, under the assumption that it is more costly to reveal more precise information. This complementary result is related to but different to the one reported in this paper. Both occur because of competition tending to eliminate the negative consequences of releasing information.

3   A Symmetric Model

An auctioneer sells a single nondivisible product to n bidders, indexed by $i \in \{1, 2, \cdots, n\}$. Bidder i's valuation of the product depends on his own taste $\tilde{\theta}_i$ as well as the product attribute $\tilde{s}$. Both $\tilde{\theta}_i$ and $\tilde{s}$ are real-valued random variables, with θ_i and s as their typical realizations respectively. Bidder i's valuation of the product is

$$v_i = u\left(s, \theta_i, \boldsymbol{\theta}_{-i}\right), \tag{1}$$

where $\boldsymbol{\theta}_{-i}$ is a vector of the tastes of all other bidders, and u is symmetric in $\boldsymbol{\theta}_{-i}$, as in the general symmetric model of MW (1982). We normalize the seller's valuation of the product to 0, and she is a revenue maximizer.

The distribution function of $\tilde{s}$ is $G(s)$, with strictly positive density $g(s)$. Bidders' tastes, $\tilde{\theta}_i$'s, are independent draws from the same distribution of $F(\theta)$, with strictly positive density $f(\theta)$, and $\tilde{s}$ and $\tilde{\theta}_i$'s are also independent from each other. The distributions of $G(s)$ and $F(\theta)$ are common knowledge, but the realized product attribute s and bidder i's taste θ_i, are respectively the seller and bidder i's private information.

Prior to the auction, the seller has the option of revealing product information to the bidders, by sending a public cheap-talk message. We focus on information structure in the

form of partitions, as common in the literature on cheap-talk. Specifically, the disclosure strategy is a mapping from the attribute space S to the message space M, denoted by $\phi : S \to M$, and for each message m, $\phi^{-1}(m)$ is a (connected) subinterval on S. We suppose M is rich enough such that ϕ can be an unto function. For a given message m, the conditional distribution of $\tilde{s}$ is denoted by $G(s \,| m)$ with the corresponding density

$$g(s \,|m) = \frac{\phi(m \,|s) \, g(s)}{\int_{x \in S} \phi(m \,|x) \, \mathrm{d}G(x)}, \tag{2}$$

and bidder i's posterior expected valuation of the product, denoted by $v(\theta_i, m)$, is thus

$$v(\theta_i, m) = \int_{s \in S} \mathbb{E}\left[u(\theta_i, \boldsymbol{\theta}_{-i}; s)\right] g(s \,|m) \, \mathrm{d}s, \tag{3}$$

based on which he submits his bids $b(\theta_i, m)$. We denote the vector of n bidders' bidding strategies as $\mathbf{b}(m) = (b(\theta_1, m), \cdots, b(\theta_n, m))$, and bidder i's optimal bidding strategy as $b^*(\theta_i, m)$ that maximizes his expected payoff in the auction. Conditional on s and the public message m, the bidders submit their bids, and we define the expected auction revenue as *interim* expected auction revenue, denoted by $R(m, s)$.

The timing of the game is as follow: first, Nature selects the realizations of product attribute $\tilde{s}$ and the bidders' tastes, $\tilde{\theta}_i$'s, which are privately observed by the seller and corresponding bidders respectively; second, based on her observation of s, the seller sends a public message m (and set a reserve price $r(s)$, if setting reserve prices is allowed); third, bidders update their beliefs and offer bids in a standard auction; finally, the auction and the final payoffs are implemented.

The equilibrium concept applied here is *perfect Bayesian equilibrium (PBE)*: i) bidders' belief follows Bayesian rule of (2), when it is applicable; ii) given the beliefs, bidders' bidding strategies are optimal; iii) the seller's disclosure strategy is also optimal. When reserve prices are not considered, the seller's strategy is to set the disclosure policy, and we define the message-only equilibrium, denoted as m-PBE, as follows.

Definition 1 *A m-PBE consists of a strategy profile $(\phi; \mathbf{b})$ and belief system $g(s \,|m)$ such that: i) $g(s \,|m)$ is derived from Bayesian rule of (2) when applicable; ii) bidders offer their optimal bids given their beliefs, that is, $b(\theta_i, m) = b^*(\theta_i, m)$ for all i; iii) the seller chooses optimal signalling strategy such that*

$$R(\phi(s), s) \geq R(\phi(s'), s), \quad \text{for } \forall s, s' \in S. \tag{4}$$

Under cheap-talk, the seller can send any possible message she likes. The incentive compatible condition of (4) implies that, in equilibrium, the seller will prefer truthful information disclosure rather than deviation. Therefore, the m-PBE constitutes an informative equilibrium where the revealed information is credible and informative.

Auction, as a selling mechanism, may involve the selection of reserve prices, which is similar to posted prices in the context of monopoly pricing. When reserve prices are considered, the seller's strategy becomes twofold: besides the signalling strategy of $\phi(s)$, she

also needs to set the reserve price $r(s)$ conditional on the realization of product attribute. With reserve prices, the interim expected auction revenue is denoted by $R(\phi(s), r(s), s)$. Similarly, we define the message-price equilibrium, denoted by (m, r)-PBE, as follows.

Definition 2 *A (m, r)-PBE consists of a strategy profile $(\phi, r; \mathbf{b})$ and belief system $g(s|m, r)$ such that: i) $g(s|m, r)$ is derived from Bayesian rule of (2) when applicable; ii) bidders offer their optimal bids given their beliefs, that is, $b(\theta_i; m, r) = b^*(\theta_i; m, r)$ for all i; iii) the seller chooses optimal signalling strategy such that*

$$R(\phi(s), r(s); s) \geq R(\phi(s'), r(s'); s), \quad \textit{for } \forall s, s' \in S.$$

The above definitions correspond to the PBEs under cheap-talk, where the interim incentive conditions, e.g. (4), must be satisfied in equilibria. If the seller is committed to truthful information disclosure, we can similarly define the corresponding m-PBE and (m, r)-PBE, by simply removing the incentive compatible condition of (4).

With this symmetric model, our interests lie in the characterization of relevant informative equilibria, and the optimal strategies of cheap talk disclosure in auctions. Here we focus on standard auctions, where the bidder offering the highest bids wins, such as first-, second-price and English auctions. And, as mentioned before, we will investigate both vertically and horizontally differentiated products, where the former is closely related to MW (1982), and the latter is more of our interests in this paper.

4 Vertically Differentiated (VD) Products

In the general symmetric model of MW (1982), the valuation function of $u(s, \theta_i, \boldsymbol{\theta}_{-i})$ is assumed to be increasing in both s and $(\theta_i, \boldsymbol{\theta}_{-i})$. This corresponds exactly to the case of VD products. We may think of s in this case as the vertical *quality* of the product, and all bidders prefer high quality to low quality product, when other things are equal. We keep the same assumption of MW (1982) here for this Section.

Assumption 1 (A1) *$u(s, \theta_i, \boldsymbol{\theta}_{-i})$ is non-decreasing in both $(\theta_i, \boldsymbol{\theta}_{-i})$ and s.*

Specifically, when we mention VD products, we mean assumption A1 is true. Under Assumption 1 and that the seller is committed to truthful disclosure, MW (1982) prove the following well-known result on optimal disclosure.

Proposition 3 (MW, 1982) *For VD products and under truthful disclosure, in standard auctions without reserve prices, revealing full information ($\phi(s) = s$) maximizes expected auction revenue.*

Compared with MW (1982), here in our symmetric model, we impose a weak version of positive affiliation between $\tilde{s}$ and $\tilde{\theta}_i$'s, by assuming that they are all independent from each other. MW (1982) further consider the effects of introducing reserve prices, and show that the introduction of reserve price, $r(s)$, may raise the expected auction revenues.

Proposition 4 (MW, 1982) *For VD products and under truthful disclosure, in standard auctions with given reserve prices, revealing full information maximizes expected revenue. For any other disclosure policy, there is a $(\phi(s), r(s))$ policy with $\phi(s) = s$ which yields higher ex-ante expected revenue.*

We now turn to the case of cheap-talk information disclosure, and focus on information structure in the form of partitions. For the VD products in our symmetric model, as implied in assumption A1, we show that a partition m-PBE does not exist. The basic intuition is that, for VD products, the seller always has incentive to announce that the product is of high quality, regardless of what the real product *quality* is. As a result, the revealed information can not be credible in equilibrium.

Proposition 5 *For VD products and under cheap-talk disclosure, in standard auctions without reserve prices, there does not exist a m-PBE in the form of partitions.*

Proof. Suppose there is a partition m-PBE with a finite number of messages, $\phi(s) \in \{m_1, m_2, \cdots, m_K\}$. Then for a message m_j, $\phi^{-1}(m_j)$ is a (connected) subinterval on the attribute space of $S \subset \mathbb{R}$. And we can order two messages $m_j < m_{j'}$, in such a way that for any $s \in \phi^{-1}(m_j)$ and $s' \in \phi^{-1}(m_{j'})$, $s < s'$. Under assumption A1, bidders' valuations u is non-decreasing in s, therefore for two messages $m_j < m_{j'}$, both $R(m_{j'}, s') > R(m_j, s')$ and $R(m_{j'}, s) > R(m_j, s)$, which violates the incentive compatible condition of (4). Hence m_j and $m_{j'}$ can not both be messages in a partition m-PBE. ∎

Proposition 6 *For VD products and under cheap-talk disclosure, in standard auctions with reserve prices: (1) there exists a full revealing (m, r)-PBE; (2) but the expected auction revenue in the full revealing (m, r)-PBE is lower than that in a babbling equilibrium with optimal reserve price.*

Proof. Under assumption A1, let $\underline{s}$ be the worse product attribute (quality), and $\underline{r}$ be the corresponding optimal reserve for it. (1) For any $s \in S$ and $s > \underline{s}$, choose the reserve price $r(s)$ such that $R(s, r(s); s) = R(\underline{s}, \underline{r}; \underline{s})$, which is always possible by continuity, since $R(s, r^*(s); s) > R(\underline{s}, \underline{r}; \underline{s})$, where $r^*(s)$ is the optimal reserve when s is revealed, and $\lim_{r \to \infty} R(s, r; s) = 0$. (2) The expected revenue under full revealing is $R(\underline{s}, \underline{r}; \underline{s})$, which is smaller than $R(\varnothing, r_\varnothing)$ by MW (1982), where $\varnothing$ means revealing no information and $r_\varnothing$ is the optimal reserve in this case. ∎

Gardete (2013) proposes a model of informative cheap-talk advertising for vertically differentiated products, and his key assumption is that different types of consumers have access to different outside options, which plays a similar role of setting different reserve prices here in our symmetric model.

5 Horizontally Differentiated (HD) Products

For HD products, the valuation $u(s, \theta_i, \boldsymbol{\theta}_{-i})$ is non-monotonic in s, and bidders of different tastes prefer different product attribute s. A standard setting for product horizontal

differentiation is that, a bidder's valuation of the product depends on the match between his own taste θ_i and the product attribute s. Therefore, u is single-peaked and may have the single-crossing property in (s, θ_i). And we adopt the leading example of quadratic valuation that possesses the above properties,

$$u(s, \theta_i) = V - \tau(s - \theta_i)^2, \tag{5}$$

where V is a commonly know vertical value of the product, and τ is a parameter measuring the degree of disutility of mismatch. Without loss of generality, we assume both $\tilde{s}$ and θ_i's are defined on the attribute space of $S = [-1, 1]$, and we introduce the following symmetry assumption on their distributions.

Assumption 2 (A2) *Both $g(s)$ and $f(\theta)$ are log-concave and symmetric to 0.*

Many distributions are log-concave, such as normal, uniform distribution and so on. The log-concavity assumption implies that the density functions of f and g are unimodal. And with the symmetry assumption, f and g have the common mean and mode of 0. Without specification, assumption A2 always holds in our analysis in Section 5 and 6.

As for the disclosure policy, we define a *J-partition* of the attribute interval S by a sequence of cutting point, $P_J = (s_0, s_1, s_2, \cdots, s_J)$, such that $-1 = s_0 < s_1 < \cdots < s_{J-1} < s_J = 1$, and $J \in \mathbb{Z}^+$ is defined as the *degree* of partition. Thus a J-partition divides S into J subintervals. Denote $\mathcal{P}_J$ the space of all J-partitions, and $\mathcal{P}_J$ is apparently a convex set, as the convex combination of any two J-partitions is still a J-partition. Let $\Delta_j = |s_j - s_{j-1}|$ be the length of the subinterval $[s_{j-1}, s_j]$, and we define an *equal partition* as a partition where all the subintervals are of equal length, denoted by $\underline{P}_J$.

Under a partition of P_J, the message space M is composed of J distinct messages, and the disclosure policy is denoted by ϕ_J, which is

$$\phi_J(s) = m_j \quad \text{iff } s \in [s_{j-1}, s_j), \, 0 \le j < J, \tag{6}$$

and for $j = J$, $\phi_J(s) = m_J$ iff $s \in [s_{J-1}, s_J]$. Given a message m_j in partition P_J, the conditional mean and variance of $\tilde{s}$ are denoted respectively by

$$\mu_j = \mathbb{E}(\tilde{s} | m_j), \quad \sigma_j^2 = \text{Var}(\tilde{s} | m_j). \tag{7}$$

We define signal precision intuitively based on the conditional variance of σ_j^2. Formally, for two messages m_j and $m_{j'}$ in partition P_J, m_j is said to be *more precise* than $m_{j'}$ iff $\sigma_j^2 < \sigma_{j'}^2$. In particular, when G follows a uniform distribution, $\sigma_j^2 = \Delta_j^2/12$, and m_j is more precise than $m_{j'}$ iff $\Delta_j < \Delta_{j'}$.

Under the symmetry assumption of f and g, we focus on partitions that are symmetric to 0, and assume $\mu_j \ge 0$ without loss of generality. For a partition P_J symmetric to 0, it can be equivalently represented by its positive cutting points, and there are two possibilities: when $J = 2K$, $K \in \mathbb{Z}^+$, then $P_J = (-s_K, , \cdots, -s_1, 0, s_1, \cdots, s_K)$, where $s_K = 1$ and we define $\Delta_1 = |s_1|$; when $J = 2K-1$, then $P_J = (-s_K, \cdots, -s_1, s_1, \cdots, s_K)$,

and we define $\Delta_1 = |2s_1|$; in both cases, for $j = 2, \cdots, K$, $\Delta_j = |s_j - s_{j-1}|$, as defined before. Then a partition of P_J that is symmetric to 0 can be equivalently represented by the sequence of positive cutting points, $P_J = (s_1, s_2, \cdots, s_K)$.

5.1 Interim Expected Auction Revenue: an Analysis

We first provide some characterizations of the interim expected revenue $R(m, s)$, where reserve prices are not considered. For horizontally differentiated products, a bidder's valuation of the product is given by (5). Given our assumption that $\tilde{s}$ and $\tilde{\theta}_i$'s are independent from each other, when a message m is announced, bidders' posterior valuations are actually independent draws from the same distribution. And the auction is in fact a standard independent private value auction, where the bidder with the highest bid wins, and the revenue is equal to the second highest valuation of the bidders. Therefore, $R(m, s)$ is equal to the expected value of the second highest $v(\theta_i, m)$'s. It will be shown later that $R(m, s)$ does not explicitly depend on s, and we then simplify the notation as $R(m)$.

A simple calculation from (3) and (5) shows that bidder i's posterior valuation

$$v(\theta_i, m_j) = V - \tau \left[\text{Var}(\tilde{s} | m_j) + (\theta_i - \mathbb{E}(\tilde{s} | m_j))^2 \right] = V - \tau \left[\sigma_j^2 + (\theta_i - \mu_j)^2 \right]. \quad (8)$$

It is interesting to find that revealing product attribute information, m_j, has two different effects on bidders' posterior valuations. One is the *universal* effect of risk reduction, that is, providing more precise signal (smaller σ_j^2) reduces the risk or mismatch for all of the bidders, and thus raises their valuations *ceteris paribus*. The other is the *idiosyncratic* effect of attribute revelation, that is, by sharpening the conditional expectation of the product attribute (μ_j), it not only drives up the valuations of some bidders whose tastes closely match μ_j, but drives down those of other bidders who find they are poorly matched.

To spare notation, we introduce a new random variable $\tilde{\beta}_i(s)$, which measures the distance between a bidder's taste and the product attribute, as follows,

$$\tilde{\beta}_i(s) = \left(\tilde{\theta}_i - s \right)^2. \quad (9)$$

We denote the distribution function of $\tilde{\beta}_i(s)$ by $H(\cdot; s)$, with the corresponding density $h(\cdot; s)$. Our first result is to show that, $H(\cdot; s)$, as a family of distributions indexed by s, can be ordered by dispersive order and first order stochastic dominance. We first provide the relevant definitions of the stochastic orders.

Definition 7 *Given two random variables $\tilde{\beta}_i(s')$ and $\tilde{\beta}_i(s'')$ with distribution functions $H(\cdot; s')$ and $H(\cdot; s'')$ respectively, we say that*

1) $\tilde{\beta}_i(s')$ is smaller than $\tilde{\beta}_i(s'')$ in the dispersive order, denoted by $\tilde{\beta}_i(s') \preceq_{disp} \tilde{\beta}_i(s'')$, if $H^{-1}(q; s') - H^{-1}(p; s') \leq H^{-1}(q; s'') - H^{-1}(p; s'')$ for all $0 < p < q < 1$.

2) $\tilde{\beta}_i(s')$ is smaller than $\tilde{\beta}_i(s'')$ in the first order stochastic dominance, denoted by $\tilde{\beta}_i(s') \preceq_{FOSD} \tilde{\beta}_i(s'')$, if $H(x; s') \geq H(x; s'')$ for all $x \in \mathbb{R}$.

Dispersive order is a stochastic order that helps to compare the variability of two

random variables, and $\tilde{\beta}_i(s') \preceq_{disp} \tilde{\beta}_i(s'')$ implies that the variance of $\tilde{\beta}_i(s')$ is smaller than that of $\tilde{\beta}_i(s'')$. Compared with the direct comparison of variances of two random variables, dispersive order provides more information on the underlying distributions. And the first order stochastic dominance is more related to the comparisons of the expectation of two random variables, and $\tilde{\beta}_i(s') \preceq_{FOSD} \tilde{\beta}_i(s'')$ implies that $\mathbb{E}\tilde{\beta}_i(s') \leq \mathbb{E}\tilde{\beta}_i(s'')$. Based on our symmetry assumption of A2 on $F(\theta)$, we have the following result.

Lemma 8 *For* $0 \leq s' < s'' \leq 1$,

  i) $\tilde{\beta}_i(s') \preceq_{disp} \tilde{\beta}_i(s'')$;

  ii) $\tilde{\beta}_i(s') \preceq_{FOSD} \tilde{\beta}_i(s'')$.

Proof. From (9), it is clear that

$$H(x;s) = F\left(s + \sqrt{x}\right) - F\left(s - \sqrt{x}\right), \qquad x \in \left[0, \bar{\beta}(s)\right], \tag{10}$$

where $\bar{\beta}(s) = (1+s)^2$. i) From result (3.B.11) of Shaked and Shanthikumar (2007), to prove $\tilde{\beta}_i(s') \preceq_{disp} \tilde{\beta}_i(s'')$, it is equivalent to show that $h\left(H^{-1}(p;s);s\right)$ is monotonically decreasing in s for all $p \in (0,1)$. Differentiating with respect to s,

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x}\frac{\partial H^{-1}}{\partial s} + \frac{\partial h}{\partial s} = -h_x\left(H^{-1}(p;s);s\right)\frac{H_s\left(H^{-1}(p;s);s\right)}{h\left(H^{-1}(p;s);s\right)} + h_s\left(H^{-1}(p;s);s\right).$$

By substituting $H(x;s)$ in (10) and $H^{-1}(p;s) = x$, we get

$$\frac{\partial h}{\partial s} = -\left[f'\left(s + \sqrt{x}\right) - f'\left(s - \sqrt{x}\right)\right]\frac{f\left(s + \sqrt{x}\right) - f\left(s - \sqrt{x}\right)}{f\left(s + \sqrt{x}\right) + f\left(s - \sqrt{x}\right)} + \left[f'\left(s + \sqrt{x}\right) + f'\left(s - \sqrt{x}\right)\right].$$

The condition for $\frac{\partial h}{\partial s} \leq 0$ appears to be equivalent to

$$\frac{f'\left(s + \sqrt{x}\right)}{f\left(s + \sqrt{x}\right)} + \frac{f'\left(s - \sqrt{x}\right)}{f\left(s - \sqrt{x}\right)} \leq 0.$$

The symmetry assumption of A2 implies that $\frac{f'\left(\sqrt{x}\right)}{f\left(\sqrt{x}\right)} + \frac{f'\left(-\sqrt{x}\right)}{f\left(-\sqrt{x}\right)} = 0$. So the above condition finally becomes

$$\frac{f'\left(\sqrt{x} + s\right)}{f\left(\sqrt{x} + s\right)} - \frac{f'\left(\sqrt{x} - s\right)}{f\left(\sqrt{x} - s\right)} \leq 0,$$

which is true given A2 that $f(\cdot)$ is log-concave and $s \geq 0$.

ii) It is clear that $H(x;s)$ is decreasing in s under A2 and $s \geq 0$, and then $\tilde{\beta}_i(s')$ is first-order stochastically dominated by $\tilde{\beta}_i(s'')$. ∎

Dispersive order and FOSD measure the variabilities and expectations of two random variables respectively. It then follows from Lemma 8 that, with increasing s, $\tilde{\beta}_i(s)$ becomes more and more dispersed with both increasing variance and increasing mean.

We next derive the expression of the interim expected auction revenue, $R(m_j)$. For a given public message m_j, $\tilde{\beta}_i(\mu_j)$'s are n independent draws from the same distribution

of $H\left(\cdot;\mu_j\right)$. The corresponding order statistics of $\tilde{\beta}_i$'s are

$$\tilde{\beta}_{1:n} \le \tilde{\beta}_{2:n} \le \cdots \le \tilde{\beta}_{n:n},$$

where $\tilde{\beta}_{k:n}$ is the kth smallest *order statistic* of the n random variables of $\tilde{\beta}_i$'s.

Apparently, $\tilde{\beta}_{1:n}$ corresponds to the bidder whose taste is the closest to μ_j, and he also has the highest posterior valuation of the product. In a standard auction, the bidder with $\tilde{\beta}_{1:n}$ wins, and his expected payment is equal to the expected valuation of the bidder with $\tilde{\beta}_{2:n}$. From (8), the interim expected auction revenue is thus

$$R\left(m_j\right) = V - \tau\left[\sigma_j^2 + \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_j\right)\right]. \tag{11}$$

Then conditional on a public message m_j, the interim expected auction revenue, $R\left(m_j\right)$, is jointly determined by signal precision, σ_j^2, and the expected value of $\tilde{\beta}_{2:n}\left(\mu_j\right)$. To characterize $R\left(m_j\right)$, we first prove some interesting properties of $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$, which are related to the results of Lemma 8. The result below shows that $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ is decreasing in n, and both increasing and convex in s.

Lemma 9 *For $s \in [0,1]$,*

i) $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ *is strictly decreasing in n, and $\lim_{n\to\infty} \mathbb{E}\tilde{\beta}_{2:n}\left(s\right) = 0$ uniformly;*

ii) $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ *is strictly increasing in s;*

iii) $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ *is strictly convex in s.*

Proof. We denote the distribution function of $\tilde{\beta}_{2:n}\left(s\right)$ by $H_{2:n}(\cdot;s)$, and the corresponding density function by $h_{2:n}(\cdot;s)$. It is easy to show that

$$H_{2:n}(x;s) = 1 - [1 - H\left(x;s\right)]^n - n\left[1 - H\left(x;s\right)\right]^{n-1} H\left(x;s\right). \tag{12}$$

i) For $x \in \left(0, \bar{\beta}\left(s\right)\right)$, $H_{2:n}(x) - H_{2:n+1}(x) = -nH\left(x\right)^2 \left[1 - H\left(x\right)\right]^{n-1} < 0$, and thus

$$\mathbb{E}\left[\tilde{\beta}_{2:n+1}\left(s\right) - \tilde{\beta}_{2:n}\left(s\right)\right] = \int_0^{\bar{\beta}(s)} \left[H_{2:n}(x) - H_{2:n+1}(x)\right] \mathrm{d}x < 0.$$

Moreover, for any $s \in [0,1]$, it is clear that

$$\lim_{n\to\infty} \mathbb{E}\tilde{\beta}_{2:n}\left(s\right) = \bar{\beta}\left(s\right) - \int_0^{\bar{\beta}(s)} \lim_{n\to\infty} H_{2:n}(x;s)\mathrm{d}x = 0$$

pointwisely. Dini's Theorem implies that $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ uniformly converges to 0 on $s \in [0,1]$.

ii) For the distribution function of $H_{2:n}(x;s)$, differentiating with respect to s, we get $\frac{\partial H_{2:n}(x;s)}{\partial s} = n\left(n-1\right)\left[1 - H\left(x\right)\right]^{n-2} H\left(x\right) \frac{\partial H(x;s)}{\partial s} < 0$ for all $x \in \left(0, \bar{\beta}\left(s\right)\right)$, as $\frac{\partial H(x)}{\partial s} < 0$ from Lemma 8. Therefore, with increasing s, the family of $H_{2:n}(x;s)$ is ordered by first-order stochastic dominance, and $\mathbb{E}\tilde{\beta}_{2:n}\left(s\right)$ is strictly increasing in s.

iii) We have $\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}}{\partial s} = -\int_0^{\bar{\beta}(s)} \frac{\partial H_{2:n}(x;s)}{\partial s} dx > 0$, where

$$\frac{\partial H_{2:n}}{\partial s} = n(n-1)(1-H)^{n-2} H \frac{\partial H}{\partial s} = \frac{\partial H_{2:n}}{\partial x} \frac{\partial H/\partial s}{\partial H/\partial x}.$$

Let $\eta(x,s) = -\frac{\partial H/\partial s}{\partial H/\partial x}$, which is strictly positive. Therefore,

$$\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}}{\partial s} = \int_0^{\bar{\beta}(s)} \eta(x,s) \, dH_{2:n}(x;s).$$

To prove $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is convex in s, it is equivalent to prove $\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}}{\partial s}$ is increasing in s. Given $H_{2:n}(x;s)$ is decreasing in s, it is sufficient to show $\eta(x,s)$ is increasing in s. We have

$$\frac{1}{2\sqrt{x}} \eta(x,s) = \frac{f(s+\sqrt{x}) - f(s-\sqrt{x})}{f(s+\sqrt{x}) + f(s-\sqrt{x})} = \frac{\frac{f(s+\sqrt{x})}{f(s-\sqrt{x})} - 1}{\frac{f(s+\sqrt{x})}{f(s-\sqrt{x})} + 1} = \frac{\frac{f(\sqrt{x}+s)}{f(\sqrt{x}-s)} - 1}{\frac{f(\sqrt{x}+s)}{f(\sqrt{x}-s)} + 1}.$$

Since $\frac{y-1}{y+1}$ is increasing, $\eta(x,s)$ is increasing in s if $\frac{f(\sqrt{x}+s)}{f(\sqrt{x}-s)}$ is decreasing in s. This is true as f is log-concave by A2. $\blacksquare$

The intuition for i) of Lemma 9 is straightforward: with increasing n, there are more and more bidders in the auction, and it then becomes more likely that bidders' tastes are on average closer to a given attribute s. Second, ii) and iii) of Lemma 9 show that, when s moves from the centre of 0 to the extreme of 1, $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is increasing at an accelerating rate. This is because, under assumption A2, when s increases from 0 to 1, the value of $f(\theta)$ gets smaller and smaller, implying that fewer bidders favor the product attribute, and therefore the average distance between θ and s gets larger. Figure 1 below provides a numerical example, where $f(\theta)$ follows a uniform distribution, that helps illustrate the properties of $\mathbb{E}\tilde{\beta}_{2:n}(s)$. The example shows that $\mathbb{E}\tilde{\beta}_{2:n}(s)$ converges to 0 when n increases, and for given n, $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is increasing and convex in s.
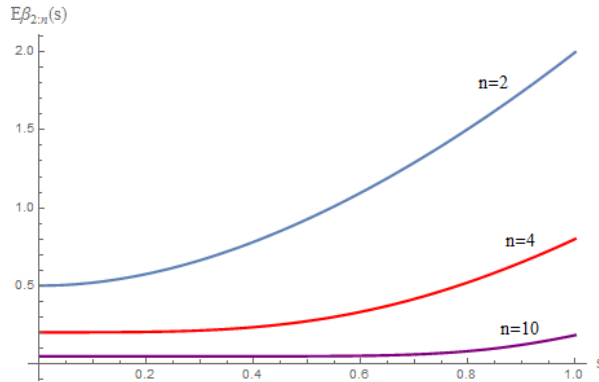

Figure 1. Properties of $\mathbb{E}\tilde{\beta}_{2:n}(s)$

The next result shows that, for two different product attributes, the increment in $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is also decreasing in n, and moreover, that increment converges to 0 uniformly when n converges to infinity.

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Lemma 10 *For $0 \leq s' < s'' \leq 1$,*

    i) $\mathbb{E}\left[\tilde{\beta}_{2:n}\left(s''\right) - \tilde{\beta}_{2:n}\left(s'\right)\right]$ *is decreasing in n;*

    ii) $\lim_{n\to\infty} \mathbb{E}\left[\tilde{\beta}_{2:n}\left(s''\right) - \tilde{\beta}_{2:n}\left(s'\right)\right] = 0$ *uniformly.*

Proof. For $0 \leq s' < s'' \leq 1$, (i) we know $\tilde{\beta}_i\left(s''\right) \succeq_{disp} \tilde{\beta}_i\left(s'\right)$, and therefore

$$X = \left[V - \tau\tilde{\beta}_i\left(s''\right)\right] \succeq_{disp} \left[V - \tau\tilde{\beta}_i\left(s'\right)\right] = Y,$$

by the property of dispersive order. From Theorem 6 of Ganuza and Penalva (2010), we have $\mathbb{E}\left[X_{2:n} - Y_{2:n}\right] \geq \mathbb{E}\left[X_{2:n-1} - Y_{2:n-1}\right]$, which is equivalent to

$$\mathbb{E}\left[\tilde{\beta}_{2:n}\left(s''\right) - \tilde{\beta}_{2:n}\left(s'\right)\right] \leq \mathbb{E}\left[\tilde{\beta}_{2:n-1}\left(s''\right) - \tilde{\beta}_{2:n-1}\left(s'\right)\right].$$

(ii) As $\lim_{n\to\infty} \mathbb{E}\tilde{\beta}_{2:n}\left(s\right) = 0$ uniformly on $s \in [0,1]$ from Lemma 9, then for any $0 \leq s' < s'' \leq 1$, $\lim_{n\to\infty} \mathbb{E}\left[\tilde{\beta}_{2:n}\left(s''\right) - \tilde{\beta}_{2:n}\left(s'\right)\right] = 0$ pointwisely. From Dini's Theorem, we have $\lim_{n\to\infty} \mathbb{E}\left[\tilde{\beta}_{2:n}\left(s''\right) - \tilde{\beta}_{2:n}\left(s'\right)\right] = 0$ uniformly. $\blacksquare$

It is worth attention that the results in Lemma 8, 9 and 10 do not depend on the specific format of a disclosure rule, e.g. whether it is partition or not. What really matters is the assumption that the distribution of bidders' tastes, F, is log-concave and symmetric to 0. We will repeatedly apply these results in our following discussion of the informative equilibria and the proof of equilibrium existence.

5.2 Informative Equilibria: Characterizations

If a m-PBE exists, then the seller's incentive compatible condition of (4) at the interim stage must be satisfied. As $R(m,s)$ does not explicitly depend on s, as shown in (11), it then follows from (4) that, in equilibrium

$$R\left(m_j\right) = R\left(m_{j'}\right), \quad \text{for } \forall s, s' \in S. \tag{13}$$

This condition implies that, in a m-PBE, all the interim expected revenues are equal. Given our symmetric setting, we just focus on partitions symmetric to 0, which can be equivalently represented by a sequence of positive cutting points. We denote the optimal partition by $P_J^* = (s_1^*, s_2^*, \cdots, s_K^*)$. For a given message m_j, μ_j and σ_j^2 denote the conditional mean and variance of the product attribute respectively. The following equilibrium property is a direct implication of the equal revenue condition of (13).

Proposition 11 *For HD products and under cheap-talk, if there exists a m-PBE partition P_J^*, then for any two distinct messages m_j and $m_{j'}$ such that $\mu_j < \mu_{j'}$, we have*

$$\sigma_j^2 > \sigma_{j'}^2, \tag{14}$$

which is equivalent to $\Delta_j > \Delta_{j'}$ if $G(s)$ is uniform distribution.

Proof. From (11), the interim expected auction revenue is $R(m_j)$ is decreasing in both σ_j^2 and μ_j, as $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is strictly increasing in s. For two distinct messages m_j and $m_{j'}$, if $\mu_j < \mu_{j'}$, then it is necessary that $\sigma_j^2 > \sigma_{j'}^2$ in equilibrium, otherwise the equal revenue condition of (13) can not be satisfied. Second, the result for uniform distribution is obvious as $\sigma_j^2 = \Delta_j^2/12$ when G is uniform. ∎

Proposition 11 states that, in a m-PBE , when the expected product attribute moves from the centre towards the extremes, it is necessary to provide more precise signals for those attributes. The intuition is as follows. Given the assumption that the distribution of bidders' tastes is unimodal and symmetric to 0, it is clear that when the product attribute moves from 0 to the extreme of 1, the product becomes less and less popular. For the less popular products, the interim expected revenue is lower *ceteris paribus*. However, in a m-PBE, it is necessary that all the $R(m_j)$'s be equal, otherwise the seller will deviate from truthful revealing. For example, she will just report the message that generates the highest interim expected revenue, regardless of the real product attribute, which makes the revealed information not credible. As a result, it is necessary to provide more precise signals for the less popular product attributes in equilibrium, so as to compensate that revenue deficit.

We could provide a formal definition on the popularity of a product attribute, as follows: a product attribute s is more popular than s', if at any positive price, the demand for s is greater than that for s'. Given our assumption of symmetric distribution of $F(\theta)$ in A2, it is evident that, when the product attribute moves from the centre of 0 to the extremes of ± 1, it becomes less and less popular. We then have the following corollary.

Corollary 12 *For HD products and under cheap-talk, in a m-PBE, the seller provides more precise information for less popular product attributes.*

Another implication of Proposition 11 is that full information disclosure is not possible under cheap-talk, when the number of bidders is given. Full information disclosure implies that the seller reports the true value of s to the bidders, and thus $\phi(s) = s$ and $\sigma_j^2 = 0$ always. However, as $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is strictly increasing in s, the equal revenue condition of (13) can never be satisfied in equilibrium, in this case.

Corollary 13 *For HD products and under cheap-talk, full information disclosure can not happen in a m-PBE, when n is given.*

This result is distinct from the full disclosure result in the literature (Ganuza, 2004; Board, 2009; Ganuza and Penalva, 2010; Hummel and McAfee, 2015), where the seller is committed to truthful disclosure. We next show that, when n converges to infinity, then the m-PBE partition will converge to a partition where all the signals are of the same precision level. This is because, with increasing n, the difference in $\mathbb{E}\tilde{\beta}_{2:n}(s)$ across different product attributes converges uniformly to 0 (Lemma 10), and therefore the equal revenue condition of (13) necessarily implies that the seller needs to provide equal precise signals for all the product attributes in equilibrium.

Corollary 14 *For HD products and under cheap-talk, in a m-PBE partition P_J^*, for any two signals m_j and $m_{j'}$, we have*

$$\lim_{n\to\infty} \left(\sigma_j^2 - \sigma_{j'}^2\right) = 0 \ for \ \forall j, j'.$$

Moreover, if G is uniform distribution, then $\lim_{n\to\infty} P_j^* = \underline{P}_J$.

Proof. The equal revenue condition of (13) implies that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \left[\left(\sigma_j^2 + \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right) - \left(\sigma_{j'}^2 + \mathbb{E}\tilde{\beta}_{2:n}(\mu_{j'})\right)\right] \\
&= \lim_{n\to\infty} \left(\sigma_j^2 - \sigma_{j'}^2\right) + \lim_{n\to\infty} \mathbb{E}\left[\tilde{\beta}_{2:n}(\mu_j) - \tilde{\beta}_{2:n}(\mu_{j'})\right] \\
&= \lim_{n\to\infty} \left(\sigma_j^2 - \sigma_{j'}^2\right) \qquad \text{[from Lemma 10 (ii)]}
\end{aligned}
$$

Second, when G is uniform, then $\lim_{n\to\infty}\left(\sigma_j^2 - \sigma_{j'}^2\right) = \frac{1}{12}\lim_{n\to\infty}\left(\Delta_j^2 - \Delta_{j'}^2\right) = 0$, which implies that $\lim_{n\to\infty} P_j^* = \underline{P}_J$. ∎

Below we provide a numerical example, where both $F(\theta)$ and $G(s)$ follow a uniform distribution on S, that helps to illustrate the properties of the m-PBEs. And the example shows that, in an equilibrium partition, more precise information is provided for less popular product attributes, and that, with increasing number of bidders, the equilibrium partition P_J^* converges to the equal partition of $\underline{P}_J$.

Example 15 *We provide an numerical example where both $F(\theta)$ and $G(s)$ follow a uniform distribution on $S = [-1, 1]$. We restrict out attention to partitions of degree 3, and denote the partition as $P_3 = (-1, -s_1, s_1, 1)$. And an equal partition corresponds to the cutting point of $s_1 = \frac{1}{3}$, apparently.*
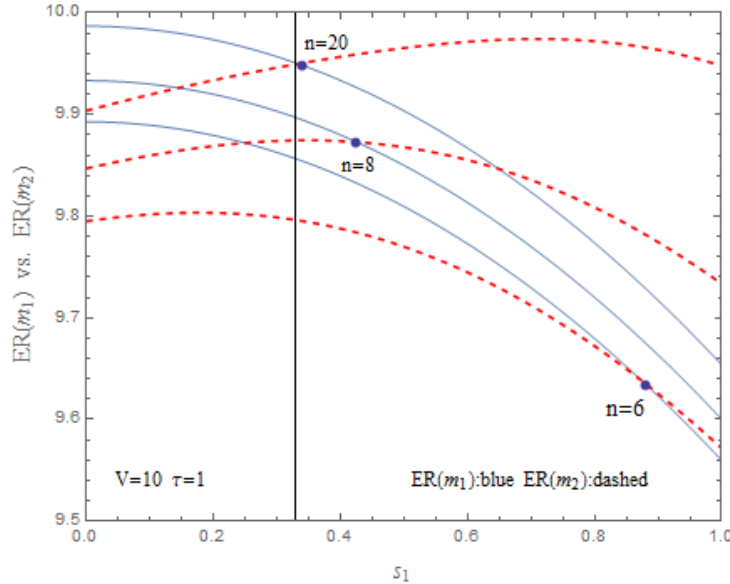

Figure 2. Properties of Informative Equilibria: an Example

First, Proposition 11 implies that, for given n, in an equilibrium partitions of P_3^*, it is necessary that $s_1^* > \frac{1}{3}$. Second, Corollary 14 implies that, when $n \to \infty$, P_3^* will converge

to equal partition, that is $\lim_{n\to\infty} s_1^* = \frac{1}{3}$.

We select $V = 10$ and $\tau = 1$. In Figure 2, the solid lines shows how the interim expected revenue $R(m_1)$ changes with s_1 for different n's. Similarly, the dashed lines shows that of $R(m_2)$. For a given n, the crossing point of $R(m_1)$ and $R(m_2)$ represents a m-PBE partition, where the equal revenue condition of (13) is satisfied. Figure 2 shows that for $n = 6, 8$ and 20, the crossing points s_1^* are all to the right of $\frac{1}{3}$, which confirms the results of Proposition 11. Second, when n increases, the crossing point moves up-left, and becomes more and more close to the equal partition point of $s_1 = \frac{1}{3}$, which confirms the convergence result of Corollary 14.

5.3 Informative Equilibria: an Existence Theorem

We have provided some characterizations of the m-PBEs, yet haven't shown such equilibria do exist. In this section, we provide an existence theorem, which shows that, for any given partition degree $J \in \mathbb{Z}^+$, there always exists a m-PBE in the form of partitions, P_J^*, as long as the number of bidders is sufficiently large. In the proof, we apply the Intermediate Value Theorem (IVT) defined on partial order sets (Guilerme, 1995), and the main objective of the proof is to show that, under certain conditions, there exists a partition such that all the interim expected revenues are equal.

The interim expected revenue, as in (11), is determined by both $\mathbb{E}\tilde{\beta}_{2:n}(\mu_j)$ and σ_j^2. Specifically, the conditional variance of σ_j^2 is related to the underlying distribution of $G(s|m_j)$. When $G(s)$ is a uniform distribution, there is a simple expression of the conditional variance, $\sigma_j^2 = \frac{1}{12}\Delta_j^2$, which is solely determined by the length of the subinterval. To make the basic intuition more clear, while at the same time avoid unnecessary complications, we make the following simplification assumption on $G(s)$. Without specification, we assume the following assumption of A3 is true for the parts of Section 5 and 6.

Assumption 3 (A3) $G(s)$ is uniform distribution on S.

The proof of the existence of m-PBEs is composed of four steps: 1) to define a metric space of partitions, $(\mathcal{P}_J, d)$, and the partial order on it, and to show that it is convex and thus connected; 2) to define a continuous vector valued function $\mathbf{f} : \mathcal{P}_J \longrightarrow \mathbb{R}^{K-1}$, with its jth element $f_j(P_J) = \frac{1}{\tau}[R(m_j) - R(m_{j-1})]$, $j = 2, \cdots, K$, which is the revenue difference of two neighboring subintervals; 3) to construct two particular partitions, $\underline{P}_J$ and $\bar{P}_J$, such that $\underline{P}_J < \bar{P}_J$, and show that for n being large enough, $\mathbf{f}(\underline{P}_J) < \mathbf{0} < \mathbf{f}(\bar{P}_J)$; 4) finally, applying the IVT, we then can prove that, if n is sufficiently large, there exists a partition P_J^*, such that $\underline{P}_J < P_J^* < \bar{P}_J$ and $\mathbf{f}(P_J^*) = \mathbf{0}$.

The IVT approach is somehow different from the standard approach of using fixed point theorems to prove equilibrium existence. As to be shown later, compared with fixed point theorems, the condition for applying the IVT is more restrictive, in the sense that it imposes more restrictions on the monotonicity of functions. However, the result is also more informative, as, different from fixed point theorems, it indicates the relative location of the equilibrium. And the proof is proceeded as follows.

First, a partition P_J symmetric to 0 can be equivalently presented by a sequence of K positive cutting points, that is, $P_J = (s_1, s_2, \cdots, s_K)$ where $s_j > 0$ and $s_K = 1$. For the partition space $\mathcal{P}_J$, we define a metric $d(\cdot, \cdot)$ in the usual way that, for any $P_J, P'_J \in \mathcal{P}_J$, $d(P_J, P'_J) = \sqrt{\sum_{j=1}^{K} \left(s_j - s'_j\right)^2}$, and we then define the metric space of $(\mathcal{P}_J, d)$. Next, we introduce a partial order, $\geq$, on $\mathcal{P}_J$ that, for any $P_J, P'_J \in \mathcal{P}_J$, $P_J \geq P'_J$ iff $s_j \geq s'_j$ for all $j = 1, \cdots, K$, and we then define the partial order set of $(\mathcal{P}_J; \geq)$. Apparently, the partition space $\mathcal{P}_J$ is convex, as any convex combination of two partitions $P_J, P'_J \in \mathcal{P}_J$ is also a partition in $\mathcal{P}_J$. Convexity implies that $\mathcal{P}_J$ is connected.

Second, we define a vector-valued function $\mathbf{f} : \mathcal{P}_J \longrightarrow \mathbb{R}^{K-1}$, with its jth element

$$f_j(P_J) = \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}\left(\mu_{j+1}\right) - \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_j\right)\right], \quad j = 1, 2, \cdots, K-1. \tag{15}$$

In fact, $f_j(P_J) = \frac{1}{\tau}\left[R(m_{j+1}) - R(m_j)\right]$, which is the difference in interim expected auction revenues of two neighboring subintervals. And $\mathbf{f}(P_J) = \mathbf{0}$ implies that all the interim expected revenues are equal.

Third, we construct two particular partitions, denoted by $\underline{P}_J = (\underline{s}_1, \underline{s}_2, \cdots, \underline{s}_K)$ and $\bar{P}_J = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_K)$, such that $\underline{P}_J < \bar{P}_J$ and when n is sufficiently large, $\mathbf{f}(\underline{P}_J) < \mathbf{0} < \mathbf{f}(\bar{P}_J)$. In fact, $\underline{P}_J$ is just the equal partition, where all the subintervals are of the same length. In an equal partition of $\underline{P}_J$, under assumption A3, we have $\sigma_j^2 = \sigma_{j'}^2$ for any j, j', which implies that $f_j(\underline{P}_J) = \mathbb{E}\left[\tilde{\beta}_{2:n}\left(\mu_j\right) - \tilde{\beta}_{2:n}\left(\mu_{j+1}\right)\right] < 0$ for all $j = 1, 2, \cdots, K-1$, as $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is strictly increasing in s from Lemma 9. We then have, for any given n

$$\mathbf{f}(\underline{P}_J) < \mathbf{0}. \tag{16}$$

Furthermore, the partition of $\bar{P}_J = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_K)$ is constructed in such a way that

$$\bar{\Delta}_j = \bar{\Delta}_{j+1} + \delta(J) \text{ for } j = 1, \cdots, K-1 \text{ and } \bar{\Delta}_K \in (0, 2/J), \tag{17}$$

where $\bar{\Delta}_j = |\bar{s}_j - \bar{s}_{j-1}|$ and $\delta(J)$ is a strictly positive term, which is fully determined by $\bar{\Delta}_K$. We next show that, by construction, $\underline{P}_J < \bar{P}_J$.

Lemma 16 *For $\underline{P}_J$ and $\bar{P}_J$, we have*

i) $\underline{P}_J < \bar{P}_J$;

ii) *for any $P_J = (s_1, s_2, \cdots, s_K)$ such that $\underline{P}_J \leq P_J \leq \bar{P}_J$,*

$$(s_{j+1} - s_j) \geq \bar{\Delta}_K \quad \text{for all } j = 1, 2, \cdots, K-1. \tag{18}$$

Proof. For the equal partition $\underline{P}_J = (\underline{s}_1, \underline{s}_2, \cdots, \underline{s}_K)$, it follows that

$$\underline{s}_j = 1 - (K - j)\frac{2}{J}, \quad j = 1, 2, \cdots, K,$$

where $J = 2K$ or $2K - 1$. And for $\bar{P}_J = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_K)$, it is easy to show that

$$\bar{s}_j = 1 - (K - j)\left[\bar{\Delta}_K + \frac{1}{2}(K - j - 1)\delta(J)\right], \qquad j = 1, 2, \cdots, K,$$

where

$$\delta(J) = \begin{cases} \frac{4}{(J-1)^2}\left(2 - J\bar{\Delta}_K\right) & \text{if } J = 2K - 1 \\ \frac{4}{(J-1)^2 - 1}\left(2 - J\bar{\Delta}_K\right) & \text{if } J = 2K \end{cases}.$$

For $\bar{P}_J$ to be non-trivial, it's needed that $K \geq 2$. We first show that for all $j < K$, we always have $\underline{s}_j < \bar{s}_j$: i) when $J = 2K$, it follows that $\bar{s}_j - \underline{s}_j = (K - j)\left(\frac{1}{K} - \Delta_K\right)\frac{j}{K-1} > 0$; ii) when $J = 2K - 1$, similarly, we have $\bar{s}_j - \underline{s}_j = \left(\frac{2}{J} - \Delta_K\right)\frac{(K-j)}{2(K-1)}\left(\frac{2K-1}{K-1}j - 1\right) > 0$. Then we show that, by construction, $\underline{P}_J < \bar{P}_J$.

Second, for any partition $P_J = (s_1, s_2, \cdots, s_K)$ satisfying $\underline{P}_J \leq P_J \leq \bar{P}_J$, we have $s_{j+1} - s_j \geq \underline{s}_{j+1} - \bar{s}_j = \kappa(j)$, which is shown to be bounded below by $\bar{\Delta}_K$: i) when $J = 2K$,

$$\kappa(j) = \frac{1}{K} - (\bar{s}_j - \underline{s}_j) = \frac{1}{K} - (K - j)\left(\frac{1}{K} - \bar{\Delta}_K\right)\frac{j}{K-1},$$

which is strictly increasing in j, so $\min_j \kappa(j, \Delta_K) = \underline{s}_2 - \bar{s}_1 = \Delta_K$; ii) when $J = 2K - 1$,

$$\kappa(j) = \frac{2}{J} - (\bar{s}_j - \underline{s}_{j+1}) = \frac{2}{J} - \left(\frac{2}{J} - \Delta_K\right)\frac{(K-j)}{2(K-1)}\left(\frac{2K-1}{K-1}j - 1\right),$$

which is again strictly increasing in j, so $\min_j \kappa(j) = \underline{s}_2 - \bar{s}_1 \geq \Delta_K$. ∎

At this stage, we have shown that $\mathbf{f}(\underline{P}_J) < \mathbf{0}$ for any given n, and that $\underline{P}_J < \bar{P}_J$ by construction. Next we will show that, when n is sufficiently large, $\mathbf{f}(\bar{P}_J) > \mathbf{0}$.

Lemma 17 *There exists an $\bar{N}_J \in \mathbb{Z}^+$ such that for any $n \geq \bar{N}_J$, $\mathbf{f}(\bar{P}_J) > \mathbf{0}$.*

Proof. From (17), the partition of $\bar{P}_J$ is fully determined by $\bar{\Delta}_K$. And it is easy to show that, in partition $\bar{P}_J$,

$$\min_j\left(\bar{\Delta}_j^2 - \bar{\Delta}_{j+1}^2\right) = \bar{\Delta}_{K-1}^2 - \bar{\Delta}_K^2 = \delta(J)\left[2\bar{\Delta}_K + \delta(J)\right] = \epsilon\left(\bar{\Delta}_K\right) > 0.$$

As $\mathbb{E}\tilde{\beta}_{2:n}(s)$ uniformly converges to 0 (Lemma 7), there exists a $\bar{N}_J \in \mathbb{Z}^+$ such that for any $n \geq \bar{N}_J$, $0 < \mathbb{E}\tilde{\beta}_{2:n}(s) < \frac{1}{12}\epsilon\left(\bar{\Delta}_K\right)$ for all $s \in [0, 1]$. Therefore, for $n \geq \bar{N}_J$, and $\forall j = 1, 2, \cdots, K - 1$,

$$f_j(\bar{P}_J) = \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right] > \frac{1}{12}\epsilon\left(\bar{\Delta}_K\right) - \frac{1}{12}\epsilon\left(\bar{\Delta}_K\right) = 0,$$

and hence, $\mathbf{f}(\bar{P}_J; n) > \mathbf{0}$. ∎

So far we have shown that, for the two particular partitions $\underline{P}_J$ and $\bar{P}_J$, by construction $\underline{P}_J < \bar{P}_J$, and when n is large enough, $\mathbf{f}(\underline{P}_J) < \mathbf{0} < \mathbf{f}(\bar{P}_J)$. To prove the existence of an m-PBE, we next show that, when n is sufficiently large, there exists an equilibrium partition, denoted by P_J^*, such that $\underline{P}_J < P_J^* < \bar{P}_J$ and $\mathbf{f}(P_J^*) = \mathbf{0}$. Here we apply the

IVT for semi-continuous functions defined on partial order sets (Guillerme,1995). The theorem is quoted as below

Theorem 18 (IVT, Guillerme, 1995) *Let* $\mathbf{h} := (h_i)_{i \in I}$ *be a function from an interval of* $[\mathbf{u}, \mathbf{v}]$ *in* $\mathbb{R}^I$ *with* $\mathbf{u} \leq \mathbf{v}$. *Suppose that, for each* i *in* I *and each* $\mathbf{x}$ *in* $[\mathbf{u}, \mathbf{v}]$, *the following properties are fulfilled:*

    (1) the function $h_i(\cdot, x_{-i})$ *is upper semi-continuous on the right on* $[u_i, v_i]$;

    (2) the function $h_i(\cdot, x_{-i})$ *is lower semi-continuous on the left on* $[u_i, v_i]$;

    (3) the function $h_i(x_i, \cdot)$ *is nonincreasing on* $[\mathbf{u}_{-i}, \mathbf{v}_{-i}]$.

    Then the interval $[\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{v})]$ *is contained in the set* $\mathbf{h}([\mathbf{u}, \mathbf{v}])$.

Compared with the standard IVT theorem for real-valued function, the IVT theorem for vector-valued functions not only requires the continuity of the function, but imposes some monotonicity restrictions on its element functions. We will show that these conditions are all naturally satisfied in our model, and we have the following existence theorem.

Theorem 19 (Existence Theorem) *For any* $J \in \mathbb{Z}^+$, *there exists a minimum number of bidders,* $N(J)$, *such that for any* $n \geq N(J)$, *a m-PBE can be supported by a J-partition.*

Proof. As shown above, the set of J-partitions, $\mathcal{P}_J$, is a convex hull in $\mathbb{R}_+^K$ and therefore connected. Next, for the defined vector-valued function $\mathbf{f} : \mathcal{P}_J \longrightarrow \mathbb{R}^{K-1}$, its jth element

$$f_j(P_J) = \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}\left(\mu_{j+1}\right) - \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_j\right)\right], \quad j = 1, 2, \cdots, K-1,$$

is continuous in $\mathcal{P}_J$ and thus conditions (1) and (2) of Guillerme (1995)'s IVT Theorem are satisfied. And by construction, we know $\underline{P}_J < \bar{P}_J$ and, for any $n \geq \bar{N}_J$,

$$\mathbf{f}\left(\underline{P}_J\right) < \mathbf{0} < \mathbf{f}\left(\bar{P}_J\right).$$

We next show that, when n is sufficiently large, the monotonicity condition of (3) is satisfied. First, $\frac{\partial f_j}{\partial s_{j+1}} = -\frac{1}{6}\Delta_{j+1} - \frac{1}{2}\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)}{\partial \mu_j} < 0$ as $\mathbb{E}\tilde{\beta}_{2:n}(s)$ is increasing in s. Second,

$$\frac{\partial f_j}{\partial s_{j-1}} = -\frac{1}{6}\Delta_j + \frac{1}{2}\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_j\right)}{\partial \mu_j},$$

where the first term is negative and the second is positive. We already know from (18) in Lemma 15, that, for any partition $P_J \in \left[\underline{P}_J, \bar{P}_J\right]$, $\Delta_j = s_j - s_{j-1} \geq \bar{\Delta}_K > 0$. Furthermore, as $\mathbb{E}\left[\tilde{\beta}_{2:n}\left(\mu_{j+1}\right) - \tilde{\beta}_{2:n}\left(\mu_j\right)\right]$ uniformly converges to 0 (Lemma 10), then there exists an $N\left(\bar{\Delta}_K\right)$ such that for any $n \geq N\left(\bar{\Delta}_K\right)$, $\frac{\partial \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)}{\partial \mu_j} < \frac{1}{3}\bar{\Delta}_K$, and thus $\frac{\partial f_j(P_J)}{\partial s_{j-1}} < 0$. Let $N(J) = \max\left\{\bar{N}_J, N\left(\bar{\Delta}_K\right)\right\}$. Then for any $n \geq N(J)$, conditions (1), (2) and (3) of the IVT are all satisfied, and there exists an equilibrium partition $P_J^* \in \left[\underline{P}_J, \bar{P}_J\right]$, such that

$$\mathbf{f}\left(P_J^*\right) = \mathbf{0}.$$

And a message-only informative equilibrium can be supported by a partition of P_J^*. ∎

Therefore, we can always find an informative equilibrium for given partition degree J, as long as the number of bidders is sufficiently large. The intuition is that, with increasing number of bidders, the differences in popularity, measured by $\mathbb{E}\tilde{\beta}_{2:n}(s)$, across different product attributes converge to zero (Lemma 10), and it becomes easier for the seller to adjust signal precision, measured by σ_j^2, to get the equal revenue condition of (13) to be satisfied. In the limit, when the number of bidders converges to infinity, the differences in popularity across product attributes converge to 0, and the equilibrium partitions will converge to partitions where all the signals are of the same level of precision. And the following corollary shows that, with increasing n, the equilibrium partitions $P_J^*(n)$ will get smaller and smaller, and gradually converge to the equal partition of $\underline{P}_J$.

Corollary 20 *For given n, if a m-PBE J-partition exists, denote as $P_J^*(n)$, then for any $n' > n$, there also exists a m-PBE J-partition, denoted as $\tilde{P}_J^*(n')$. Moreover,*

$$\underline{P}_J < \tilde{P}_J^*(n') < P_J^*(n),$$

and $\lim_{n \to \infty} P_J^*(n) = \underline{P}_J$.

Proof. Suppose in a n bidder auction, the J-partition of $P_J^*(n) = (s_1^*, s_2^*, s_3^*, \cdots, s_K^*)$ supports a m-PBE. Then we have, for any $j = 1, 2, \cdots, K - 1$

$$f_j(P_J^*(n)) = \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right] = 0.$$

For $n' > n$, in the same J-partition $P_J^*(n)$, we have

$$
\begin{aligned}
f_j\left(P_J^*(n')\right) &= \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n'}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n'}(\mu_j)\right] \\
&> \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right] = 0,
\end{aligned}
$$

because $\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)$ is strictly decreasing in n, from Lemma 4. Therefore, $\mathbf{f}(P_J^*) > \mathbf{0}$, and we still have $\mathbf{f}(\underline{P}_J) < \mathbf{0}$. By applying the IVT on poset again, it follows that there exists a J-partition $\tilde{P}_J^*(n') \in [\underline{P}_J, P_J^*(n)]$ such that $\mathbf{f}\left(\tilde{P}_J^*(n')\right) = \mathbf{0}$. The convergence result of $\lim_{n \to \infty} P_J^*(n) = \underline{P}_J$ is self-evident, given the property in Lemma 10 that $\lim_{n \to \infty} \left[\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right] = 0$. ∎

The corollary shows that, when the number of bidders increases, the equilibrium partition $\tilde{P}_J^*(n)$ gets smaller and smaller, and gradually converges to the equal partition of $\underline{P}_J$ when n converges to infinity. The intuition is straightforward. For an equilibrium partition, we know from (13) that all the interim expected revenues are equal, which are jointly determined by signal precision δ_j^2 and $\mathbb{E}\tilde{\beta}_{2:n}(\mu_j)$. With increasing n, the differences in $\mathbb{E}\tilde{\beta}_{2:n}(\mu_j)$'s across different messages converge to 0 (Lemma 10), and therefore the differences in signal precision also need to decrease, such that the equal revenue condition continues to be held in the new equilibrium $\tilde{P}_J^*(n')$. In consequence, with increasing n, the equilibrium partitions become smaller and smaller and under assumption A3, converge to the equal partition of $\underline{P}_J$.

Another result is that, when the number of bidders is given, the maximum possible degree for an m-PBE partition is bounded above. This result, as Corollary 13, also implies that full information disclosure is not achievable in an m-PBE, when the number of bidders is given. Formally, the result is as below.

Corollary 21 *For given n, there exists a maximum degree for m-PBE partitions, denoted as $\bar{J}(n)$. Furthermore, $\bar{J}(n)$ is nondecreasing in n.*

Proof. The first part is just the contrapositive of the existence theorem, and the second part is implied by Corollary 20. ∎

Corollary 21 shows that for given n, there exists an upper bound for the possible degree of m-PBE partitions. On the other hand, the Existence Theorem above shows that, for given J, there exists an equilibrium J-partition as long as the number of bidders n is sufficiently large. Next, we will show that, for given n, if there exists an equilibrium partition of degree J, then for any positive integral $J' < J$, there also exists an equilibrium partition of degree J'. The proof is similar as before, we first show that for $J' = J - 1$ or $J - 2$, there exists an equilibrium J'-partition, and then the result is proved by mathematics induction.

As before, we construct two particular partitions, $\underline{P}_{J'}$ and $\bar{P}_{J'}$, where $\underline{P}_{J'}$ is an equal partition of degree J'. Let $P_J^* = (s_1^*, s_2^*, \cdots, s_K^*)$ be an equilibrium partition when there are n bidders, and $\bar{P}_{J'}$ is constructed as follows. First, for $J' = J - 2$, $\bar{P}_{J-2}$ is constructed by removing the two cutting points of $\pm s_1^*$, and it is easy to show that $\underline{P}_{J-2} < \bar{P}_{J-2}$. Second, for $J' = J - 1$, there are two possible cases: i) if $J = 2K$, then $\bar{P}_{J-1}$ is constructed by removing the cutting point of $s_0^* = 0$; ii) if $J = 2K - 1$, then $\bar{P}_{J-1}$ is constructed by removing the two cutting points of $\pm s_1^*$ and introducing a new cutting point of $s_0^* = 0$. It is clear that $\underline{P}_{J-1} < \bar{P}_{J-1}$ in both cases.

Proposition 22 *For given n, if there exists a m-PBE partition of degree J, then for any $J' < J$, there also exist an equilibrium partition of degree J', denoted as $P_{J'}^*$. Furthermore, $\underline{P}_{J'} < P_{J'}^* < \bar{P}_{J'}$.*

Proof. For $J' < J$, we define the vector-valued function, $\mathbf{f} : \mathcal{P}_{J'} \longrightarrow \mathbb{R}^{K'-1}$, with its jth element being

$$f_j(P_{J'}) = \left(\sigma_j^2 - \sigma_{j+1}^2\right) - \left[\mathbb{E}\tilde{\beta}_{2:n}(\mu_{j+1}) - \mathbb{E}\tilde{\beta}_{2:n}(\mu_j)\right],$$

for $j = 1, 2, \cdots, K' - 1$. We also denote the two particular partitions by $\underline{P}_{J'}$ and $\bar{P}_{J'}$ respectively. As before, $\underline{P}_{J'}$ is the equal partition of degree J', and it is clear that $\mathbf{f}(\underline{P}_{J'}) < \mathbf{0}$. We will consider the cases of $J' = J - 2$ and $J' = J - 2$, and $\bar{P}_{J'}$ will be constructed in the following way.

Let $P_J^* = (s_1^*, s_2^*, s_3^*, \cdots, s_K^*)$ be the equilibrium J-partition when there are n bidders, and denote the corresponding conditional means and variances under m_j by σ_j^2 and μ_j respectively. In the first case of $J' = J - 2$, $\bar{P}_{J-2}$ is constructed by removing the cutting

point of s_1^*, and therefore $\bar{P}_{J-2} = (\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_{K-1}) = (s_2^*, s_3^*, \cdots, s_K^*)$. It then follows that

$$
\begin{aligned}
f_1\left(\bar{P}_{J-2}\right) &= \left(\bar{\sigma}_1^2 - \bar{\sigma}_2^2\right) - \mathbb{E}\left[\tilde{\beta}_{2:n}\left(\bar{\mu}_2\right) - \tilde{\beta}_{2:n}\left(\bar{\mu}_1\right)\right] \\
&= \left(\bar{\sigma}_1^2 - \sigma_2^2\right) - \mathbb{E}\left[\tilde{\beta}_{2:n}\left(\mu_2\right) - \tilde{\beta}_{2:n}\left(\bar{\mu}_1\right)\right] \\
&> \left(\sigma_1^2 - \sigma_2^2\right) - \mathbb{E}\left[\tilde{\beta}_{2:n}\left(\mu_2\right) - \tilde{\beta}_{2:n}\left(\mu_1\right)\right] = 0,
\end{aligned}
$$

as $\bar{\sigma}_1^2 > \sigma_1^2$ and $\bar{\mu}_1 > \mu_1$. And for other $j > 1$, $f_j\left(\bar{P}_{J-2}\right) = 0$ as P_J^* is an equilibrium partition. Therefore, $\mathbf{f}\left(\bar{P}_{J'}\right) \geq \mathbf{0}$.

In the second case of $J' = J - 1$, if $J = 2K$, then the partition $\bar{P}_{J'}$ is constructed by removing the cutting point of $s_0^* = 0$ in P_J^*, and we then have

$$
\bar{P}_{J-1} : -1 = -s_K^* < \cdots < -s_1^* < s_1^* < \cdots < s_K^* = 1.
$$

In this new partition of $\bar{P}_{J-1}$, $\bar{\mu}_1 = 0$ and

$$
\begin{aligned}
& \mathbb{E}\tilde{\beta}_{2:n}\left(\bar{\mu}_K\right) - \mathbb{E}\tilde{\beta}_{2:n}\left(\bar{\mu}_1\right) \\
=\ & \mathbb{E}\tilde{\beta}_{2:n}\left(\bar{\mu}_K\right) - \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_1\right) + \mathbb{E}\tilde{\beta}_{2:n}\left(\mu_1\right) - \mathbb{E}\tilde{\beta}_{2:n}\left(\bar{\mu}_1\right) \\
<\ & 2\mathbb{E}\left[\tilde{\beta}_{2:n}\left(\mu_K\right) - \tilde{\beta}_{2:n}\left(\mu_1\right)\right] \quad (\mathbb{E}\tilde{\beta}_{2:n} \text{ convex in } \mu\ \&\ \bar{\mu}_K - \mu_1 > \mu_1) \\
=\ & 2\left(\sigma_1^2 - \sigma_K^2\right) \qquad\qquad\quad (\bar{\mu}_K = \mu_K^*\ \&\ P_J^* \text{ is equilibrium partition}) \\
<\ & \left(\bar{\sigma}_1^2 - \sigma_K^2\right) \qquad\qquad\quad (\bar{\sigma}_1^2 = 4\sigma_1^2\ \&\ \sigma_1^2 > \sigma_K^2)
\end{aligned}
$$

and thus $f_1\left(\bar{P}_{J-1}\right) > 0$. Similarly, from the equilibrium condition, we have for $j > 1$, $f_j\left(\bar{P}_{J-2}\right) = 0$, and therefore $\mathbf{f}\left(\bar{P}_{J'}\right) \geq \mathbf{0}$ as before. Following the similar reasoning, we can also show that it is also true for the case of $J = 2K - 1$. Therefore, we have $\underline{P}_{J'} < \bar{P}_{J'}$, $\mathbf{f}\left(\underline{P}_{J'}\right) < \mathbf{0}$ and $\mathbf{f}\left(\bar{P}_{J'}\right) \geq \mathbf{0}$ for $J' = J - 1$ and $J - 2$. By the Intermediate Value Theorem, there exists a partition of degree J', $P_{J'}^*$, such that $\mathbf{f}\left(P_{J'}^*\right) = \mathbf{0}$. Finally, by mathematical induction, we get the result. ∎

5.4 Optimal Disclosure Policy

We next investigate the optimal disclosure policy for the seller. We know that in a m-PBE, all the interim expected revenues are equal, and therefore the ex ante expected auction revenue in equilibrium is just equal to the interim ones. We denote the ex ante expected auction revenue in a J-degree m-PBE partition equilibrium by $R(J)$.

The first question is when revealing information would be profitable, if compared with revealing no information at all. In our model, when the partition degree $J = 1$, it is equivalent to revealing null information. The question is then when $R(J) > R(1)$ for $J \geq 2$. The advantage of withholding information is that it won't change bidders' *ex ante* expectation of the product attribute, which is 0, the most popular attribute. However, not revealing product information also implies that bidders face the largest risk of mismatch between their tastes and the exact product attribute. So the combined effect on expected auction revenue is mixed. The result below shows that, when the number of bidders is large enough, it is profitable for the seller to reveal product information.

Lemma 23 *There exists a $\hat{N}$ such that, for any $n \geq \hat{N}$, revealing product information generates higher expected auction revenue than withholding information.*

Proof. When there's no information disclosure, $J = 1$ and the mean and variance of the product attribute in this case is $\mu = 0$ and $\sigma^2 = \int_{-1}^{1} x^2 \mathrm{d}G(x)$. Therefore, the expected revenue is

$$R(1) = V - \tau \left[\sigma^2 + \mathbb{E}\tilde{\beta}_{2:n}(0) \right].$$

We just need to show that a symmetric partition of $J = 2$ generates higher expected revenue than $R(1)$ when n is large enough. When $J = 2$, then the conditional mean and variance of product attribute is $\mu_1 = \int_0^1 x \mathrm{d}G(x \,|\, x \geq 0) > 0$ and $\sigma_1^2 = \int_0^1 x^2 \mathrm{d}G(x \,|\, x \geq 0) - \mu_1^2 < \sigma^2$. And the expected revenue is

$$R(2) = V - \tau \left[\sigma_1^2 + \mathbb{E}\tilde{\beta}_{2:n}(\mu_1) \right].$$

As $\mathbb{E}\left[\tilde{\beta}_{2:n}(\mu_1) - \tilde{\beta}_{2:n}(0) \right]$ uniformly converges to zero from Lemma 10, then there exists an $\hat{N}$ such that, for any $n \geq \hat{N}$, $0 < \mathbb{E}\left[\tilde{\beta}_{2:n}(\mu_1) - \tilde{\beta}_{2:n}(0) \right] < \sigma^2 - \sigma_1^2$, which implies that $R(2) > R(1)$. ∎

When both $G(s)$ and $F(\theta)$ follow uniform distribution on S, it follows that $\hat{N} = 3$. So in this case, when $n \geq 3$, it is better for the seller to reveal product information to the bidders, rather than withhold it. A related yet different result is reported in Board (2009), in a reduced-form model. Board (2009) shows that, when bidders' valuation distribution is symmetric, the seller is indifferent between revealing product information or not when $n = 3$. This is because, when $n = 3$, the expectation of the second highest valuation is just equal to the ex ante expectation of bidders' valuations, and as a result, revealing information has no impacts on expected auction revenue. The uniform example of our model shows that, in contrast, when $n = 3$, the seller strictly prefers revealing product information, rather than not.

We next turn to a general result of the optimal disclosure policy. The existence theorem states that, for any given $J \in \mathbb{Z}^+$, there exists an equilibrium partition of degree J, as long as the number of bidders, n, is sufficiently large. On the other hand, for given n, there exists a maximum degree for equilibrium partition, denoted as $\bar{J}(n)$, and for any partition degree $J' \leq \bar{J}(n)$, there also exists an equilibrium partition of degree J'. And $\bar{J}(n)$ is non-decreasing in n.

Roughly speaking, a higher degree equilibrium partition corresponds to finer partition of the attribute space, and thus more precise signals. And more precise signals correspond to smaller conditional variances of the product attribute, which raise bidders' posterior valuations. For given n, let $J^*(n)$ denote the optimal degree of m-PBE partitions, at which the seller achieves the highest ex ante expected auction revenue. Obviously $J^*(n) \leq \bar{J}(n)$. The following result shows that when n is large enough, $J^*(n) = \bar{J}(n)$.

Theorem 24 *When n is large enough, the optimal degree of m-PBE partition $J^*(n)$ is*

equal to the maximum possible degree of m-PBE partitions, that is,

$$J^*(n) = \bar{J}(n) \tag{19}$$

Proof. For given n, let us consider two degrees of equilibrium partitions, $\hat{J}$ and $\bar{J}(n)$ with $\hat{J} < \bar{J}(n)$. In equilibrium partition $P^*_{\bar{J}}$, the minimum length of its subintervals is $\Delta_{\bar{K}} = |1 - s_{\bar{K}-1}|$, where $\bar{J} = 2\bar{K}$ or $2\bar{K} - 1$. Obviously, $\Delta_{\bar{K}} < 2/\bar{J}$, and the ex ante expected auction revenue is equal to

$$R(\bar{J}) = V - \tau \left[\frac{1}{12} \Delta_{\bar{K}}^2 + \mathbb{E}\tilde{\beta}_{2:n}(\mu_{\bar{K}}) \right].$$

In equilibrium partition $P^*_{\hat{J}}$, the maximum length of its subintervals is $\Delta_1 = \tilde{s}_1$ if $\hat{J} = 2\hat{K}$, or $\Delta_1 = 2\hat{s}_1$ if $\hat{J} = 2\hat{K} - 1$. And we have $\Delta_1 > 2/\hat{J}$ as well, and the ex ante expected auction revenue is equal to

$$R(\hat{J}) = V - \tau \left[\frac{1}{12} \Delta_1^2 + \mathbb{E}\tilde{\beta}_{2:n}(\hat{\mu}_1) \right],$$

and we get

$$
\begin{aligned}
\frac{1}{\tau} \left[R(\bar{J}(n)) - R(\hat{J}) \right] &= \left[\frac{1}{12}\Delta_1^2 + \mathbb{E}\tilde{\beta}_{2:n}(\hat{\mu}_1) \right] - \left[\frac{1}{12}\Delta_{\bar{K}}^2 + \mathbb{E}\tilde{\beta}_{2:n}(\mu_{\bar{K}}) \right] \\
&> \frac{1}{3}\left(\frac{1}{\hat{J}^2} - \frac{1}{\bar{J}^2} \right) - \mathbb{E}\left[\tilde{\beta}_{2:n}(\mu_{\bar{K}}) - \tilde{\beta}_{2:n}(\hat{\mu}_1) \right],
\end{aligned}
$$

which is strictly positive for sufficiently large n, as $\mathbb{E}\left[\tilde{\beta}_{2:n}(\mu_{\bar{K}}) - \tilde{\beta}_{2:n}(\hat{\mu}_1) \right]$ uniformly converges to 0 when n converges to infinity. ∎

The result states that, when n is large enough, it is optimal to select the disclosure policy corresponding to the largest possible degree of equilibrium partitions, that is, $\bar{J}(n)$. The intuition behind this result is similar to that behind the Existence Theorem. When the number of bidders increases, the difference in popularity across different product attributes becomes more and more negligible, and signal precision becomes the principle determinant for auction revenue. In this case, it is better to provide more precise product information to the bidders.

We already know that $\bar{J}(n)$ is non-decreasing in n. Then the optimal disclosure policy then shows a complementarity relationship between the number of bidders and the degree of optimal equilibrium partitions. Roughly speaking, it implies that when the number of bidders increases, it is better to reveal more precise production information to the bidders. This complementarity result looks similar to the results reported in Ganuza (2004) and Ganuza and Panelva (2010), who also find that when there are more bidders in an auction, it is better for the seller to reveal more precise information. However, their results are derived under the assumption of costly information, that is, it is more costly to reveal more precise information. If information is costless, that complementarity result disappears, and the optimal disclosure policy will again be extreme, just as in Lewis and Sappington (1994), Johnson and Myatt(2006) in monopoly pricing, and Board (2009) and Hummel

McAfee (2015) in auction context.

The requirement of n to be large enough is non-trivial, as it is possible for $J^*(n) < \bar{J}(n)$ when n is small. One example is implied in the previous result of Lemma 23. For example, under our symmetric setting, for any number of n, there always exists an equilibrium partition of degree $J = 2$. If both $F(\theta)$ and $G(s)$ are uniform distributions, then when $n < 3$, $J^*(n) = 1$ from Lemma 23, which is strictly smaller than $\bar{J}(n) = 2$. Below we provide a numerical example where both F and G are uniform.

Example 25 *Suppose both product attribute and bidders' types are uniform distribution on $S = [-1, 1]$. And we can derive the exact results on optimal and maximum equilibrium partition degree, $J^*(n)$ and $\bar{J}(n)$, following similar methods as in Example 15. We conduct our calculation till $n = 22$, and the results are as follows*

$$
J^*(n) = \begin{cases} 1 & \text{if } n \in \{1, 2\} \\ 2 & \text{if } n \in \{3, \cdots, 7\} \\ 3 & \text{if } n \in \{8, \cdots, 11\} \\ 4 & \text{if } n \in \{12, \cdots, 17\} \\ 5 & \text{if } n \in \{18, \cdots, 21\} \\ 6 & \text{if } n = 22 \end{cases}, \qquad \bar{J}(n) = \begin{cases} 2 & \text{if } n \in \{1, 2 \cdots, 5\} \\ 3 & \text{if } n \in \{6, \cdots, 11\} \\ 4 & \text{if } n \in \{12, \cdots, 17\} \\ 5 & \text{if } n \in \{18, \cdots, 21\} \\ 6 & \text{if } n = 22 \end{cases}.
$$

The numerical result shows that $J^*(n)$, the optimal degree of m-PBE partition, is non-decreasing in n. And when $n \geq 8$, $J^*(n) = \bar{J}(n)$, as shown in Theorem 24. Figure 3. below provides an illustration of the numerical results.
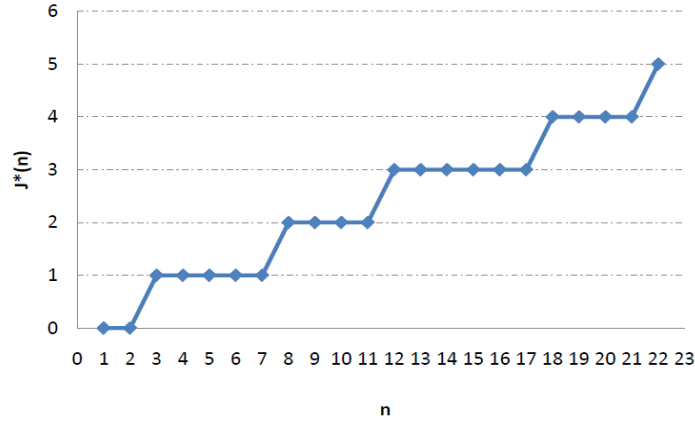

Figure 3. an Example of Optimal Information Disclosure

5.5 Informative Equilibria: with Reserve Prices

As shown in Corollary 13, in a standard auction without reserve prices, when n is given, full information disclosure is not achievable in a m-PBE. Following a similar reasoning as in Proposition 6, we next show that, if reserve prices are allowed, there full revealing can be achieved in (m, r)-PBE. The intuition is that, with the additional price instrument, the seller can adjust the reserve prices such that all the interim expected revenues be equal. And that interim expected revenue is equal to that for the worst attribute $(s = 1)$

26

with optimal reserve. A different result from Proposition 6 for VD products is that, in this case of HD products, when n is sufficiently large, full revealing equilibrium generates higher expected auction revenue than that for null information disclosure.

Proposition 26 *For HD products and under cheap-talk, when setting reserve prices is allowed in standard auctions,*

   i) *There exist fully revealing (m, r)-PBE;*

   ii) *If n is sufficiently large, full revealing (m, r)-PBE generates higher expected auction revenue than null disclosure.*

Proof. i) The extreme attribute of $s = 1$ is the worst attribute, let $r^*(1)$ be the seller's optimal reserve price for that attribute. For any other attribute $s' \in S$, there exists a reserve price $r(s)$ such that $R(1, r^*(1); 1) = R(s', r(s'); s')$, which is always possible as $R(1, r^*(1); 1) \leq R(s', r^*(s'); s')$ and $\lim_{r(s') \to V} R(s', r(s'); s') = 0$. ii) Under null disclosure, the expected second highest valuation of the bidders is $V - \tau \left[\sigma^2 + \mathbb{E}\tilde{\beta}_{2:n}(0) \right]$ which is bounded below $V - \tau \sigma^2$. Under full information disclosure, the expected second highest valuation of the worst attribute $s = 1$ is $V - \tau \mathbb{E}\tilde{\beta}_{2:n}(1)$. When $n \to \infty$, $V - \tau \mathbb{E}\tilde{\beta}_{2:n}(1)$ converges to V with probability 1, and therefore, in an auction with optimal reserve prices, full revealing equilibrium generates higher expected revenue than that of null disclosure. ∎

6 Horizontally Differentiated Products: Truthful Disclosure

For horizontally differentiated products, we now turn to the case where the seller is committed to truthful information disclosure. Similar questions have been studied in Ganuza (2004), Board (2009), Ganuza and Penalva (2010) and Hummel and McAfee (2015), but the key difference here is that we re-investigate it in a model with endogenous valuations of the bidders. Our main result is that, in this case of full commitment, when n is large enough, it is optimal for the seller to reveal full information.

When the seller is committed to a preset disclosure rule, such as (7), the incentive compatible of (4) at the interim stage is absent in equilibrium. The seller's problem is thus to maximize the ex ante expected auction revenue $R(J)$ which is

$$R(J) = \sum_{j=1}^{J} \Pr(m_j) R(m_j) = \mathbb{E}_{\tilde{m}}[R(m)], \qquad (20)$$

where $\tilde{m}$ is the random variable of signal and m is its typical realization.

Proposition 27 *For HD products and under truthful disclosure, when n is sufficiently large, full information disclosure is optimal.*

Proof. Applying the rule of total variance, we have

$$\text{Var}(\tilde{s}) = \text{Var}_{\tilde{m}}[\mathbb{E}[\tilde{s}|\tilde{m}]] + \mathbb{E}_{\tilde{m}}[\text{Var}(\tilde{s}|\tilde{m})] = \text{Var}_{\tilde{m}}[\tilde{\mu}] + \mathbb{E}_{\tilde{m}}[\tilde{\sigma}^2],$$

where $\tilde{\mu} = \mathbb{E}\left(\tilde{s}\,|\tilde{m}\,\right)$ and $\tilde{\sigma}^2 = \mathrm{Var}\left(\tilde{s}\,|\tilde{m}\,\right)$, and $\mathrm{Var}_{\tilde{m}}\left[\tilde{\mu}\right] = \mathbb{E}_{\tilde{m}}\left[\tilde{\mu}^2\right]$. Therefore,

$$\mathbb{E}_{\tilde{m}}\left[\tilde{\sigma}^2\right] = \mathrm{Var}\left(\tilde{s}\right) - \mathbb{E}_{\tilde{m}}\left[\tilde{\mu}^2\right].$$

Substituting back to the expression of $R\left(m\right)$ in (11), we then have

$$
\begin{aligned}
R &= \mathbb{E}_{\tilde{m}}\left[R\left(m\right)\right] = V - \tau\mathbb{E}_{\tilde{m}}\left[\tilde{\sigma}^2 + \mathbb{E}_{\tilde{s}}\left[\beta_{2:n}\left(\tilde{\mu}\right)\right]\right] \\
&= V - \tau\mathrm{Var}\left(\tilde{s}\right) + \tau\mathbb{E}_{\tilde{m}}\left[\tilde{\mu}^2 - \mathbb{E}_{\tilde{s}}\left[\beta_{2:n}\left(\tilde{\mu}\right)\right]\right].
\end{aligned}
$$

As $\lim_{n\to\infty}\mathbb{E}_{\tilde{s}}\left[\beta_{2:n}\left(s\right)\right] = 0$ uniformly, then the function $s^2 - \mathbb{E}_{\tilde{s}}\left[\beta_{2:n}\left(s\right)\right]$ is convex when n is sufficiently large. And from Jasen's inequality, R achieves its maximum under full revealing, when n is sufficiently large. ∎

This result of full information disclosure is also reported in Board(2009) and Ganuza and Penalva (2010) in the reduced-form models, where there is no explicit matching between preference and product attribute. It is also worth noting that the result of full disclosure in our model does not depend on the specific format of disclosure rules, whether it is partition or not. We next prove a convergence property of the optimal partition $\hat{P}_J^*$, as below.

Proposition 28 *For HD products and under truthful disclosure, when $n \to \infty$, the optimal partition of $\hat{P}_J^*$ satisfies*

$$\hat{s}_j^* = \frac{1}{2}\left(\hat{\mu}_j + \hat{\mu}_{j+1}\right), \quad for\ j = 1, 2, \cdots, K-1.$$

Specifically, when $G\left(s\right)$ is uniform, $\lim_{n\to\infty}\hat{P}_J^* = \underline{P}_J$.

Proof. We have, when n converges to infinity,

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\Pr\left(m_j\right)R\left(m_j\right) + \Pr\left(m_{j+1}\right)R\left(m_{j+1}\right)\right] \\
&= \left(G\left(s_{j+1}\right) - G\left(s_{j-1}\right)\right)V - \tau\left[\int_{s_{j-1}}^{s_j}\left(x - \mu_j\right)^2 g\left(x\right)\mathrm{d}x + \int_{s_j}^{s_{j+1}}\left(x - \mu_{j+1}\right)^2 g\left(x\right)\mathrm{d}x\right].
\end{aligned}
$$

Furthermore, as $\lim_{n\to\infty}R\left(m_j\right) = V - \tau\sigma_j^2$ uniformly, then $\lim_{n\to\infty}\frac{\partial R}{\partial s_j} = \frac{\partial}{\partial s_j}\left[\lim_{n\to\infty}R\right]$. It follows that the first order condition

$$-\frac{1}{\tau}\lim_{n\to\infty}\frac{\partial R}{\partial s_j} = \left[\left(s_j - \mu_j\right)^2 - \left(s_j - \mu_{j+1}\right)^2\right]g\left(s_j\right) = 0.$$

As $g\left(s_j\right) > 0$, we then get $2\hat{s}_j^* = \hat{\mu}_j + \hat{\mu}_{j+1}$. It is easy to show that $\hat{s}_j^* \leq \frac{1}{2}\left(\hat{s}_{j-1}^* + \hat{s}_{j+1}^*\right)$ under A2. Finally, when G is uniform, $\mu_j = \frac{1}{2}\left(s_{j-1} + s_j\right)$ and the optimal partition $\hat{P}_J^*$ then converges to equal partition. ∎

7 Concluding Remarks

This paper contributes to a large existing literature on information disclosure in auctions but it is among the first, and only, to study cheap talk information disclosure (Li (2012)). We investigate how an informed seller may reveal information to bidders through partitional cheap-talk, prior to a standard auction. We show that this cheap sales talk has quite different implications for VD and HD products. In the case of VD products, we re-examine the general symmetric model of MW (1982) and confirm that seller optimal cheap talk equilibria convey no information to bidders whether or no sellers may also set reserve prices.

In the case of HD products, we prove that, for given partition degree J, a message-only informative equilibrium can always be supported by a partition of degree J, as long as the number of bidders is sufficiently large. On the other hand, we also show that, for given number of bidders n, there exists a maximum partition degree $\bar{J}(n)$, below which a message-only informative equilibrium can be supported, and that $\bar{J}(n)$ is non-decreasing in n. Equilibria have the feature that more precise information is revealed for less popular product attributes. We show that the seller optimal disclosure policy displays a complementarity between the number of bidders and the optimal amount of product information disclosed to the bidders.

We also consider the impacts of setting reserve prices on the equilibrium outcomes, and show that in contrast to the case of VD products, a full-revealing equilibrium for HD products can result in higher revenue level than null information with optimal reserve price, as long as the number of bidders is sufficiently large.

The paper also introduces what might prove to be a useful methodological tool: in the proof of the existence theorem, we adopt the Intermediate Value Theorem defined on partially ordered sets (Guillerme, 1995), rather than a fixed point theorem.

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